Translation invariant subspaces of weighted Hilbert spaces of sequences

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Abstract

A map $\omega : \mathbb{Z} \to (0, +\infty)$ is called a weight if it satisfies the condition

$$0 < \inf_{n\in\mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} < \sup_{n\in\mathbb{Z}} \frac{\omega(n+1)}{\omega(n)} < +\infty. \quad (1)$$

In this situation the shift operator $S_\omega : (u_n)_{n\in\mathbb{Z}} \to (u_{n-1})_{n\in\mathbb{Z}}$ is bounded and invertible on the weighted Hilbert space

$$l^2(\omega) := \left\{ u = (u_n)_{n\in\mathbb{Z}} \ \mid \ \|u\|_\omega := \left[\sum_{n\in\mathbb{Z}} |u_n|^2 \omega^2(n)\right]^{1/2} < +\infty \right\}.$$

A closed subspace M of $l^2(\omega)$ is said to be *translation invariant* if $S_\omega(M) \cap S_\omega^{-1}(M) \subset M$ or, equivalently, if $S_\omega(M) = M$, and M is said to be *nontrivial* if $M \neq \{0\}$ and $M \neq l^2(\omega)$. The existence of a nontrivial translation invariant subspace of $l^2(\omega)$ for all weights ω is an open problem (in a major breakthrough, A. Atzmon showed that the answer is positive if $\omega(-n) = \omega(n)$ for $n \geq 1$). If we set $\omega^*(n) = 1/\omega(-n)$ for $n \geq 1$, it is easily seen that this problem is equivalent to the existence of sequences $u \in l^2(\omega) \setminus \{0\}$ and $v \in l^2(\omega^*) \setminus \{0\}$ satisfying the convolution equation $u * v = 0$. We will discuss three aspects of this problem.

1. We will describe Domar's elegant positive answer to the analogous question for the real line
2. We will interpret in terms of the so-called Brown approximation scheme a construction of Apostol which provides rich families of nontrivial translation invariant subspaces for "irregular" weights, and point out the obstructions which prevented so far this method to provide nontrivial translation invariant subspaces for all weights for which the spectrum of the shift operator (which is always a circle or an annulus centered at the origin) has nonempty interior
3. Using the theory of almost analytic functions in the unit disc and the theory of "Dynkin extensions" we will provide a complete classification of the translation invariant subspaces of $l^2(\omega)$ for weights ω satisfying $\omega(n) = 1$ for $n \geq 0$ and $\omega(n) = e^{\frac{|n|}{log(|n|+1)^a}}$ for $n < 0$, wih $a > 1$. We will also relate for a large class of "dissymetric" weights the translation invariant subspace problem to a problem concerning zero-free z- invariant subspaces for some Hardy space of holomorphic functions on the open unit disc.