Harmonic analysis for ρ-contractions

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Abstract

Recall that an operator T acting on a Hilbert space H admits a *unitary ρ-dilation ($\rho > 0$)*, in the sense of B. Sz. Nagy and C. Foias, if there exists a Hilbert space $\mathcal{H}$ containing H as a closed subspace and a unitary operator U on $\mathcal{H}$ such that

$$T^n = \rho P_H U^n \mid H \qquad (n \geq 1),$$

where P_H is the orthogonal projection from $\mathcal{H}$ onto H. We denote by $C_\rho(H)$ the set of all operators on H which admit a unitary ρ-dilation and we write $C_\infty(H) = \bigcup_{\rho 0} C_\rho(H)$. A famous result of B. Sz. Nagy tells us that $C_1(H)$ consists of all contractions on H and C. A. Berger proved in a paper entitled "*A strange dilation theorem*" that $C_2(H)$ consists of all operators T acting on H with numerical radius less or equal to one. Any operator $T \in C_\rho(H)$ will be referred to as a *ρ-contraction* acting on H. The study of operators belonging to $C_\infty(H)$ was elarged upon by many authors in several directions.

In the first part of this course, we recall many properties and basic tools used in the study of ρ-contractions.

In the second part, we firstly give constrained von Neumann inequalities for the ρ-numerical radius and applications to inequalities of coefficients of rational functions positive on the torus.

In the third part, we introduce an Harnack ordering between ρ -contractions and define the corresponding Harnack parts in the class C_ρ. This part involves some operatorial Harnack inequalities which generalize the Harnack inequalities for the classical contractions given by C. Foias, K. Fan and I. Suciu. We also define the Harnack and hyperbolic metrics and establish useful properties for these metrics. We recall that a famous result of C. Foias tells us that a Banach space E, where the class of contractions is stable under the Möbius field, is necessarly isometric to a Hilbert space. This property is a very useful tool for the study of contractions. We will see that the other classes $C_\rho(H)$ $(\rho \neq 1)$ are not stable under the Möbius field. What can be said about membership of $f(T)$ in the classes $C_\rho(H)$ when T belongs to a given one of them and f is a holomorphic self-map of the unit disc which is continuous on the torus? We will answer this question.

The purpose of the fourth part is to give some sharpened forms of the von Neumann inequality for strict ρ-contactions. In particular, we recover the results of K. Fan covering the strict contractions and in the scalar context we find an improved form of the interior Schwarz inequality quoted by R. Osserman. Also, some sharpened forms of the Schwartz inequality for ρ-contactions will be given, and as applications, corresponding inequalities for strict contractions and uniformly stable operators will be derived.

The last part is devoted to establish a useful mapping formula for fonctional calculus associated with ρ-contactions, the properties of the dilatation operator radius and a general Julia's lemma for operators whose spectrum is contained in the closed unit disc. A application to the hyperbolic metric on the Harnack parts of ρ-contactions is given.

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