

# A Survey of Weighted Composition Operators on Hilbert Spaces of Analytic Functions

Carl C. Cowen

## Abstract

### I. Introduction, Background, and Basic Results

We consider a Hilbert space of functions analytic in a domain in  $C^N$ , for example, the Hardy Hilbert space on the unit disk,  $H^2(D)$ . If  $\psi$  is a complex valued function on the domain and  $\varphi$  is an analytic map of the domain into itself, the weighted composition operator  $W_{\psi,\varphi}$  is defined by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$$

for  $z$  in the domain and  $f$  a function from the Hilbert space.

Over the past three decades, it has become increasingly apparent that weighted composition operators arise in important ways in a variety of contexts. For example, in 1964, Frank Forelli showed that every isometry of  $H^p(D)$ , for  $1 < p < \infty$  but  $p \neq 2$ , is a weighted composition operator.

The first talk will review some basic results from the theory of multiplication (Toeplitz) operators and from the theory of composition operators and use them to give basic properties of weighted composition operators, including results from the thesis of Gajath Gunatillake, on questions of boundedness, compactness, spectra, and some older results concerning semi-groups of weighted composition operators.

### II. Hermitian Weighted Composition Operators on the Hardy and Bergman Spaces

While the self-adjoint composition operators are trivial and not very interesting, there are interesting weighted composition operators that are Hermitian. These operators will be characterized and we give complete descriptions of the spectral theory of the Hermitian weighted composition operators on  $H^2$  and the Bergman space  $A^2$  (work of Eung Il Ko, Gajath Gunatillake, and the speaker).

### III. Connections and Challenges Related to Weighted Composition Operators

Weighted composition operators arise in function theoretic operator theory in many contexts and the study of these operators gives rise to interesting unsolved problems. In work a few years ago, Gallardo-Gutierrez and the speaker introduced multiple valued weighted composition operators to solve a problem concerning composition operators. However, applications of these operators (without being described in general), as well as standard weighted composition operators, date back further, at least to the study of commutants of analytic Toeplitz operators. Some of these connections and some open problems concerning weighted composition operators will be explored.