Quantization of the universal Teichmüller space

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Abstract

In the first part of the course we describe Kähler geometry of the universal Teichmüller space $\mathcal{T}$. This space consists of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. orientation-preserving homeomorphisms of S^1, extending to quasiconformal homeomorphisms of the unit disc Δ), normalized modulo Möbius transformations (i.e. fractional-linear automorphisms of Δ). The space $\mathcal{T}$ is a complex Banach manifold, its complex structure being induced by its embedding into the Banach space of holomorphic quadratic differentials in the unit disc Δ. The classical Teichmüller spaces, associated with compact Riemann surfaces of a finite genus, are embedded into $\mathcal{T}$ as complex submanifolds. The space $\mathcal{S}$ of diffeomorphisms of the unit circle S^1, normalized modulo Möbius transformations, may be considered as a smooth part of $\mathcal{T}$. Quasisymmetric homeomorphisms of S^1 act (by change of variable) on the Sobolev space V of half-differentiable functions on S^1 by bounded symplectic operators. Respectively, diffeomorphisms of S^1 act on V by Hilbert–Schmidt symplectic operators.

The second part of the course is devoted to the quantization of the universal Teichmüller space. We explain first how to quantize the smooth part $\mathcal{S}$ of $\mathcal{T}$. We use for that an embedding of $\mathcal{S}$ into the Hilbert–Schmidt Siegel disc, generated by the action of diffeomorphisms of S^1 as Hilbert–Schmidt symplectic operators on V. We construct a holomorphic Fock bundle over the Siegel disc together with a projective action of the Hilbert–Schmidt symplectic group. The infinitesimal version of this action yields a Dirac quantization of $\mathcal{S}$. Unfortunately, this method does not work for the whole universal Teichmüller space $\mathcal{T}$, since it contains non-smooth homeomorphisms of S^1. For quantization of $\mathcal{T}$ we use another approach, based on the following idea. We still have an action of quasisymmetric homeomorphisms on V but this action has no infinitesimal limit, as in the case of $\mathcal{S}$. However, following Connes, we can define a quantized infinitesimal version of this action. This quantized action generates a Connes quantization of $\mathcal{T}$.

THE PROGRAM OF THE COURSE

I. UNIVERSAL TEICHMÜLLER SPACE

1. Definition of universal Teichmüller space $\mathcal{T}$.
2. Complex and Kähler structures of $\mathcal{T}$.
3. Embedding of classical Teichmüller spaces into $\mathcal{T}$.
4. Grassmann realization of $\mathcal{T}$.

II. QUANTIZATION

5. Dirac quantization.
6. Fock space and Heisenberg representation.
7. Symplectic group action on Fock spaces.
8. Quantum calculus and Connes quantization.
9. Quantization of universal Teichmüller space.