An $\mathcal{H}_\infty^-$-calculus motivated from system theory

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In system theory it is well-known that the Toeplitz operator $M_g : L^2(0, \infty) \to L^2(0, \infty)$ with symbol $g$, which is bounded analytic on the left half plane $\mathbb{C}^-$, maps exponentials to exponentials,

$$M_g(e^{at}) = g(a)e^{at}$$  \hspace{1cm} (1)

for fixed $a < 0$. Obviously, $g \mapsto g(a)$ is a homomorphism from $\mathcal{H}_\infty^-$, the Banach algebra of bounded analytic functions on $\mathbb{C}^-$, to $\mathbb{C}$. Our idea is to replace the exponential by the strongly continuous semigroup $e^{At} = T(t)$ on the Banach space $X$. In fact, we show that the formally defined function

$$y(t) = M_g(T(\cdot)x_0)(t)$$

can be seen as the output of the linear system

$$\begin{align*}
\dot{x}(t) &= Ax(t), \\
y(t) &= C_g x(t)
\end{align*}$$

for some (unbounded) operator $C_g$. Thus, formally $y(t) = C_g T(t)x_0$. This means that $C_g$ takes the role of $g(a)$ in (1). Hence, the task is to find $C_g$ given the output mapping $x_0 \mapsto y(t)$. Incorporating the notion of admissibility, this can be done uniquely, see [2]. This construction yields an (unbounded) functional calculus. Moreover, $g(A)$ is bounded from $X_1 = (D(A), \|\cdot\|_A)$ to $X$ and weakly admissible, i.e.

$$\int_0^\infty \|y, g(A)T(t)x\|_{X'}^2 dt \leq K_A \|g\|_\infty^2 \|x\|^2 \|y\|_{X'}^2, \quad x \in D(A), y \in X'.$$

Furthermore, if $X$ is a Hilbert space, for every positive time $t$, $g(A)T(t)$ can be extended to a bounded operator with norm less than $\gamma_A t^{-1/2}$, with $\gamma_A$ independent of $t$. Finally, we give sufficient conditions for a bounded calculus, such as exact observability for Hilbert spaces (or exact observability by direction in the general case), [1, 3].

References


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