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Error estimates of optimal order in a fractional-step scheme for the $3D$ Navier-Stokes equations.

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Abstract

We present some improvements on the error estimates obtained by J.Blasco and R.Codina [2, 3] for a viscosity-splitting in time scheme, with finite element approximation, applied to the $3D$ Navier-Stokes equations. The key is to obtain new error estimates for the discrete in time derivative of velocity, which let us to reach, in particular, error of order one (in time and space) for the pressure approximation.

1 Introduction

We consider the unsteady, incompressible Navier-Stokes equations in a bounded domain $\Omega \subset I\!R^3$:

$$(P) \qquad \left\{ \begin{array}{l} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \ \text{ in } \Omega\times(0,T), \\ \mathbf{u} = 0 \ \text{ on } \partial\Omega\times(0,T), \quad \mathbf{u}_{|t=0} = \mathbf{u}_0 \ \text{ in } \Omega, \end{array} \right.$$

where $\mathbf{u}(\mathbf{x},t) \in I\!R^3$ is the fluid velocity at position $\mathbf{x}\in\Omega$ and time $t\in(0,T)$, $p(\mathbf{x},t)\in I\!R$ the pressure, $\nu > 0$ the viscosity (which is assumed constant) and $\mathbf{f}(\mathbf{x},t)\in I\!R^3$ the external force.

The main (numerical) difficulties of this problem are the coupling between the pressure and the incompressibility condition and the nonlinearity of the convective terms.

Fractional step methods in time are widely used to approximate the problem (P). They allow us to separate the effects of different operators appearing in the problem.

The origin of these methods is generally credited to the works of Chorin [4] and Teman [10]. They developed the well known projection method, where the idea is to split the convection and diffusion terms to the incompressibility constraint and its Lagrange multiplier the pressure, avoiding the computation of the Stokes problem. It is a two step scheme: first step is a Dirichlet-elliptic problem for an intermediate velocity and the second one is a free divergence projection step equivalent to a Neumann-elliptic problem for the pressure.

The main drawback of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the pressure of the scheme verifies "artificial" boundary conditions. The convergence of this projection method, was proved in [11] for the semidiscrete scheme and in [5] for a fully discrete scheme (when periodic boundary conditions are considered).

Error estimates for projection methods can be seen in [8], [9] for time discrete schemes and in [6] for a fully discrete scheme, with finite elements.

In this paper, we will study a viscosity-splitting scheme, introduced and studied by J.Blasco and R.Codina [1, 2, 3]. It is a two-step scheme, where the main numerical difficulties of (P) (namely, the treatment of nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and the relation between incompressibility $\nabla\cdot\mathbf{u} = 0$ and pressure), are split into two different steps, considering the diffusive terms in both steps.

Notice that both type of schemes, projection and viscosity-splitting schemes, can be jointly used, because the second step of the viscosity-splitting scheme could be computed with a projection method.

Considering a (regular) partition of $[0,T]$ of diameter $k = T/M$: $\{t_m = mk\}_{m=0}^M$, for a given vector $u = (u^m)_0^M$ with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k\sum_{m=0}^M \|u^m\|_X^2\right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\dots,M}\|u^m\|_X$$

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

This paper is organized as follows. In Section 2, we present first the semi-discrete in time scheme (which solution is denoted by $\mathbf{u}^{m+1/2}$ and $(\mathbf{u}^{m+1}, p^{m+1})$) and then the fully discrete scheme (which solution is denoted by $\mathbf{u}_h^{m+1/2}$ and $(\mathbf{u}_h^{m+1}, p_h^{m+1})$). We also hav the choice for the finite element spaces and their approximation properties. Finally, some known results and the main objectives of this paper are presented.

In Section 3, we introduce the problems verified by the semi-discrete in time errors and we will obtain new $O(k^{1/2})$ error estimates for $\mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}$ and $\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}$ in $l^\infty(\mathbf{H}^1)\cap l^2(\mathbf{H}^2)$ and for $e_p^m = p(t_m) - p^m$ in $l^2(H^1)$. Previous error estimates will be used to obtain $O(k^{1/2})$ in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ for the discrete in time derivative of $\mathbf{e}^{m+1/2}$ and $\mathbf{e}^{m+1}$, which are applied to get $O(k)$ for the discrete in time derivative of $\mathbf{e}^{m+1}$, in $l^2(\mathbf{L}^2)$ and in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, where a constraint on the first step of the scheme is imposed in the last case (in fact, these two estimates are obtained independently). As a consequence, the improvement of the pressure error estimates to order $O(k)$ in $l^2(L^2)$ and in $l^\infty(L^2)$ hold.

In Section 4, we will obtain new error estimates for the fully discrete scheme. We describe the problems verified by the discrete errors (comparing semi-discrete in time scheme and fully discrete scheme). Firstly, we will consider the $O(h)$ error estimates for $\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}$ and $\mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}$ in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$, which implying an improve for the estimates of the discrete velocities in the $W^{1,6}(\Omega)$-norm whether $h^2/k \le C$.

Afterwards, the $O(h)$ error estimates for the discrete in time derivative of $\mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ and in $l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)$ are obtained independently, where again a constraint on the first step of the scheme must be imposed in the last case. Moreover, $O(k+h^2)$ error estimates for $\mathbf{e}_d^{m+1}$ in $l^2(\mathbf{L}^2)$ is deduced. Finally, the pressure error estimates of order $O(h)$ in $l^2(L^2)$ and in $l^\infty(L^2)$ hold.

2 Description of the scheme and known results

2.1 Temporal Discretization

For simplicity, we consider the uniform partition of the time interval $[0,T]$ with time step $k=T/M$, $\{t_m = m\,k\}_{m=0}^M$. Given $(\mathbf{f}^m)_{m=1}^M$ an approximation of $\mathbf{f}(t_m)$ we define $(\mathbf{u}^m,p^m)_{m=1}^M$ an approximation of the solution $(\mathbf{u},p)$ of (P) in $t=t_m$ by means of the following first order splitting in time scheme:

Initialization: $\mathbf{u}^0=\mathbf{u}_0$

Time step $m+1$:

Substep 1: Given $\mathbf{u}^m$, to find $\mathbf{u}^{m+1/2}$ solution of:

$$(S_1)^{m+1}\qquad \left\{\begin{array}{rcl} \dfrac{\mathbf{u}^{m+1/2}-\mathbf{u}^m}{k}+(\mathbf{u}^m\cdot\nabla)\mathbf{u}^{m+1/2}-\nu\Delta\mathbf{u}^{m+1/2} &=& \mathbf{f}(t_{m+1}),\\ \mathbf{u}^{m+1/2}|_{\partial\Omega} &=& 0. \end{array}\right.$$

Substep 2: Given $\mathbf{u}^{m+1/2}$, to find $\mathbf{u}^{m+1}$ and p^{m+1} solution of:

$$(S_2)^{m+1}\qquad \left\{\begin{array}{rcl} \dfrac{\mathbf{u}^{m+1}-\mathbf{u}^{m+1/2}}{k}-\nu\Delta(\mathbf{u}^{m+1}-\mathbf{u}^{m+1/2})+\nabla p^{m+1} &=& 0,\\ \nabla\cdot\mathbf{u}^{m+1} &=& 0,\\ \mathbf{u}^{m+1}|_{\partial\Omega} &=& 0. \end{array}\right.$$

Adding $(S_1)^{m+1}$ and $(S_2)^{m+1}$, one has the following relation, which can be interpreted as a consistency relation:

$$(S_3)^{m+1}\qquad \left\{\begin{array}{rcl} \dfrac{\mathbf{u}^{m+1}-\mathbf{u}^m}{k}+(\mathbf{u}^m\cdot\nabla)\mathbf{u}^{m+1/2}-\nu\Delta\mathbf{u}^{m+1}+\nabla p^{m+1} &=& \mathbf{f}(t_{m+1}),\\ \nabla\cdot\mathbf{u}^{m+1} &=& 0,\\ \mathbf{u}^{m+1}|_{\partial\Omega} &=& 0. \end{array}\right.$$

2.2 Spatial discretization

We use a first order finite element approximation. Let Ω be a $3D$ polyhedron (or a $2D$ polygon) such that Stokes problem in Ω has $\mathbf{H}^2\times H^1$ regularity for velocity and pressure respectively.

We consider a family of finite dimensional spaces $\mathbf{V}_h\subset\mathbf{H}_0^1(\Omega)$ and $Q_h\subset L_0^2(\Omega)$ defined from finite element methods associated to a family of regular triangulations of the domain Ω with mesh size h. $\mathbf{V}_h$ and Q_h are thus required to satisfy:

- the Brezzi-Babuska stability condition: $\inf_{q_h \in Q_h \setminus \{0\}} \left(\sup_{\mathbf{v} \in \mathbf{V}_h \setminus \{0\}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\| \, |q_h|} \right) \geq \beta > 0.$

- the approximating properties:

$$\frac{1}{h} \inf_{\mathbf{v}_h \in \mathbf{V}_h} |\mathbf{u} - \mathbf{v}_h| + \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\| + \inf_{q_h \in Q_h} |p - p_h| \leq C\, h \, \|(\mathbf{u}, p)\|_{H^2 \times H^1}$$

We use the notation $|\cdot|$ and $(\cdot,\cdot)$ as the norm and the inner product in L^2 and $\|\cdot\|$ as the norm in H^1_0.

The fully discrete scheme is described as follows:
Initialization: Let $\mathbf{u}_h^0 \in \mathbf{V}_h$ be an approximation of $\mathbf{u}_0$

Step of time $m+1$:
Subtep 1: Given $\mathbf{u}_h^m \in \mathbf{V}_h$, to compute $\mathbf{u}_h^{m+1/2} \in \mathbf{V}_h$ such that, for each $\mathbf{v}_h \in \mathbf{V}_h$:

$$(S_1)_h^{m+1} \quad \left\{ \left(\frac{\mathbf{u}_h^{m+1/2} - \mathbf{u}_h^m}{k}, \mathbf{v}_h\right) + c(\mathbf{u}_h^m, \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla\, \mathbf{u}_h^{m+1/2}, \nabla\, \mathbf{v}_h) = (\mathbf{f}^{m+1}, \mathbf{v}_h), \right.$$

where $c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \{(\mathbf{w} \cdot \nabla \mathbf{u}, \mathbf{v}) - (\mathbf{w} \cdot \nabla \mathbf{v}, \mathbf{u})\}/2$.
Substep 2: Given $\mathbf{u}_h^{m+1/2}$, to compute $(\mathbf{u}_h^{m+1}, p_h^{m+1}) \in \mathbf{V}_h \times Q_h$ such that, for each $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$:
$(S_2)_h^{m+1}$

$$\left\{ \begin{array}{rcl} \frac{1}{k}(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}, \mathbf{v}_h) + (\nabla(\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m+1/2}), \nabla \mathbf{v}_h) - (p_h^{m+1}, \nabla \cdot \mathbf{v}_h) & = & 0, \\ (\nabla \cdot \mathbf{u}_h^{m+1}, q_h) & = & 0. \end{array} \right.$$

With respect to the effective computation of this scheme, in each time step, we have:

1. $(S_1)_h^{m+1}$ as three discrete linear convection-diffusion equations.
2. $(S_2)_h^{m+1}$ as a discrete Stokes problem.

2.3 Known Results and Objectives

Assuming the following regularity for the exact solution $(\mathbf{u}, p)$ of problem (P):

$$(R1) \qquad \mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V}),\ p \in L^\infty(H^1),\ \mathbf{u}_t \in L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1),\ \mathbf{u}_{tt} \in L^2(\mathbf{V}')$$

where $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1_0 \ :\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} = 0 \text{ on } \partial\Omega\ \})$, and the constraints on discrete parameters:

$$h^2 \leq C\, k,$$

the following error estimates hold ([1, 3]):

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(L^2) \cap l^2(H^1)} \leq C\,(k + h) \quad \text{and} \quad \|p(t_m) - p^m\|_{l^2(L^2)} \leq C\,\sqrt{k}$$

Moreover, the estimate $\|p^m - p_h^m\|_{l^2(L^2)} \leq C\, h/\sqrt{k}$ can be obtained with similar arguments.

The objectives of this work are:

1. To improve the order of error estimate in pressure, from $O(\sqrt{k}+h/\sqrt{k})$ to $O(k+h)$.
2. To improve the norm of error estimates in velocity and pressure, concretely from $l^\infty(L^2)$ to $l^\infty(H^1)$ in velocity and from $l^2(L^2)$ to $l^\infty(L^2)$ in pressure.
3. To improve the order of error estimate in velocity in norm $L^2(\mathbf{L}^2)$, from $O(k+h)$ to $O(k+h^2)$

The main result of this paper can be written as follows. Assuming the following additional regularity hypotheses:

$$(R2) \qquad \left\{ \begin{array}{l} p_t \in L^2(H^1),\ \mathbf{u}_t \in L^\infty(\mathbf{H}^1) \cap L^2(\mathbf{H}^2),\ \mathbf{u}_{tt} \in L^2(\mathbf{L}^2) \cap L^\infty(\mathbf{H}^{-1}) \\ \mathbf{u}_{ttt} \in L^2(\mathbf{V}'),\ \sqrt{t}\mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1}) \end{array} \right.$$

then,

$$\|p(t_m) - p_h^m\|_{l^2(L^2)} \le C\,(k+h), \quad \|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^2(\mathbf{L}^2)} \le C\,(k+h^2).$$

Moreover, assuming the following hypothesis for initial step:

$$|(\mathbf{u}(t_1) - \mathbf{u}^1) - (\mathbf{u}(t_0) - \mathbf{u}^0)| \le C\,k^2, \quad |(\mathbf{u}^1 - \mathbf{u}_h^1) - (\mathbf{u}^0 - \mathbf{u}_h^0)| \le C\,k\,h$$

then

$$\|\mathbf{u}(t_m) - \mathbf{u}_h^m\|_{l^\infty(\mathbf{H}^1)} \le C\,(k+h), \quad \|p(t_m) - p_h^m\|_{l^\infty(L^2)} \le C\,(k+h).$$

Therefore, we have that this scheme has the same analytical results than Euler's type schemes, improving their numerical treatment (since the main difficulties are split). In the following two sections, we will present an outline of the proof (see [7] for a complete explanation of the results).

Unfortunately, in order to assure the additional regularity hypotheses $(R2)$, it is necessary to assume that $\mathbf{u}_t(0) \in \mathbf{H}^1$, which implies a non local compatibility condition for the data $\mathbf{u}_0$ and $\mathbf{f}$ ([12]). In this sense, we could relax it approximating the first step with several auxiliary initial steps with a sufficiently small time step. Then, the approximate solutions obtained from these preliminary steps could serve as initial data for our fractional step algorithm at subsequent time steps.

3 Error estimates for the time discrete scheme

We decompose the total error in their temporal and spatial parts, introducing the corresponding time discrete scheme.

The following notations will used for the time discrete errors in $t = t_{m+1}$:

$$\mathbf{e}^{m+1/2} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1/2}, \quad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} = p(t_{m+1}) - p^{m+1},$$

and for the discrete in time derivatives of errors

$$\delta_t\,\mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \qquad \delta_t\,\mathbf{e}^{m+1/2} = \frac{\mathbf{e}^{m+1/2} - \mathbf{e}^{m-1/2}}{k}.$$

These discrete in time errors verify the following problems:

$$(E_1)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1/2} - \mathbf{e}^m) - \Delta \mathbf{e}^{m+1/2} = -\nabla\, p(t_{m+1}) + \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \\ \mathbf{e}^{m+1/2}|_{\partial\Omega} = 0, \end{array} \right.$$

where

$$\mathcal{E}^{m+1} = -\frac{1}{k}\int_{t_m}^{t_{m+1}} (t-t_m)\,\mathbf{u}_{tt}(t)\,dt - \left(\int_{t_m}^{t_{m+1}} \mathbf{u}_t\cdot\nabla\right)\mathbf{u}(t_{m+1})$$

is the consistency error, and

$$\mathbf{NL}^{m+1} = -\Big(\mathbf{e}^m\cdot\nabla\Big)\mathbf{u}(t_{m+1}) - (\mathbf{u}^m\cdot\nabla)\mathbf{e}^{m+1/2} = -\Big(\mathbf{e}^m\cdot\nabla\Big)\mathbf{u}^{m+1/2} - (\mathbf{u}(t_m)\cdot\nabla)\mathbf{e}^{m+1/2}$$

are residual terms depending of the quadratic terms. On the other hand,

$$(E_2)^{m+1} \qquad \left\{\begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}) - \Delta(\mathbf{e}^{m+1}-\mathbf{e}^{m+1/2}) - \nabla p^{m+1} = 0 \quad \text{in } \Omega \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = 0. \end{array}\right.$$

Adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$, we arrive at:

$$(E_3)^{m+1} \qquad \left\{\begin{array}{l} \delta_t\mathbf{e}^{m+1} - \Delta\mathbf{e}^{m+1} + \nabla e_p^{m+1} = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1}|_{\partial\Omega} = 0. \end{array}\right.$$

Theorem 3.1 *The following error estimate holds (for k small enough):*

$$\|e_p^{m+1}\|_{l^2(L^2)} \le C\,k.$$

Outline of the proof: It is based on the following three steps:

1. H^2 error estimates. Using the $H^2\times H^1$-regularity of Stokes problem verified by $(\mathbf{e}^{m+1}, e_p^{m+1})$ (passing in $(E_3)^{m+1}$ the term $\delta_t\mathbf{e}^{m+1}$ at the left hand side) and the H^2-regularity of the Poisson-Dirichlet problem verified by $\mathbf{e}^{m+1/2}$ (passing in $(E_3)^{m+1}$ the term $\dfrac{1}{k}(\mathbf{e}^{m+1/2}-\mathbf{e}^m)$ at the left hand side), one can prove that

$$\mathbf{e}^{m+1/2},\ \mathbf{e}^{m+1} \text{ are bounded in } l^\infty(\mathbf{H}^2)$$

2. Making $(\delta_t(E_1)^{m+1}, \delta_t\mathbf{e}^{m+1/2}) + (\delta_t(E_2)^{m+1}, \delta_t\mathbf{e}^{m+1})$, using a discrete Gronwall's lemma jointly with the proof for the initial step $\|\delta_t\mathbf{e}^1\|_{\mathbf{L}^2} \le C\,k^{1/2}$, one can prove:

$$\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t\mathbf{e}^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k^{1/2}$$

3. Duality argument. Making $(\delta_t(E_3)^{m+1}, A^{-1}\,\delta_t\mathbf{e}^{m+1})$, A being the Stokes operator, one can prove

$$\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{V}')\cap l^2(\mathbf{L}^2)} \le C\,k,$$

for k small enough (which becomes from to apply the generalized discrete Gronwall's lemma jointly with the proof for the initial step $\|\delta_t\mathbf{e}^1\|_{\mathbf{V}'} \le C\,k$)

Theorem 3.2 *Assuming $|\delta_t\mathbf{e}^1| \le C\,k$, the following error estimates hold*

$$\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \le C\,k \quad \textit{and} \quad \|e_p^{m+1}\|_{l^\infty(L^2)} \le C\,k.$$

Outline of the proof: It is based on the error estimate $\|\delta_t\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k$, obtained by making $(\delta_t(S_3)^{m+1}, \delta_t\mathbf{e}^{m+1})$. Notice that in this case, the proof for the initial step is not clear.

4 Error estimates for the spatial discretization

We define the spatial discrete errors:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$$

Then, the problems verified by these errors are:

$$(E_1)_h^{m+1} \qquad \left\{ \frac{1}{k}(\mathbf{e}_d^{m+1/2} - \mathbf{e}_d^m, \mathbf{v}_h) + (\nabla \mathbf{e}_d^{m+1/2}, \nabla \mathbf{v}_h) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \right.$$

where $\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h)$, and

$$(E_2)_h^{m+1} \qquad \left\{ \begin{array}{l} \frac{1}{k}(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}, \mathbf{v}_h) + (\nabla (\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}), \nabla \mathbf{v}_h) \\ \qquad -(e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h \\ (\nabla \cdot \mathbf{e}_d^{m+1}, q_h) = 0, \quad \forall q_h \in Q_h \end{array} \right.$$

Adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$, one has for each $(\mathbf{v}_h, q_h) \in (\mathbf{V}_h, Q_h)$:

$$(E_3)_h^{m+1} \qquad \left\{ \begin{array}{l} (\delta_t \mathbf{e}_d^{m+1}, \mathbf{v}_h) + (\nabla \mathbf{e}_d^{m+1}, \nabla \mathbf{v}_h) - (e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \\ (\nabla \cdot \mathbf{e}_d^{m+1}, q_h) = 0. \end{array} \right.$$

Theorem 4.1 *For k small enough, the following error estimate holds:*

$$\|\mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \le C(k + h^2)$$

Outline of the proof: It is based on the following steps:

1. Making $(\delta_t(E_1)_h^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}) + (\delta_t(E_2)_h^{m+1}, \delta_t \mathbf{e}_h^{m+1})$, one can arrives at

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C.$$

2. The following additional scheme estimate is obtained:

$$\mathbf{u}_h^{m+1} \text{ is bounded in } l^\infty(\mathbf{W}^{1,3} \cap \mathbf{L}^\infty). \tag{1}$$

3. Duality argument. Making $((E_3)_h^{m+1}, A_h^{-1}\, \mathbf{e}_h^{m+1})$, A_h being the discrete Stokes operator.

Theorem 4.2 *For k small enough, the following estimate holds*

$$\|e_{p,d}^{m+1}\|_{l^2(L^2)} \le C\, h.$$

Outline of the proof: It is based on the following estimate

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \le C\, h,$$

which is obtained by making $(\delta_t(E_3)_h^{m+1}, A_h^{-1}\, \delta_t \mathbf{e}_h^{m+1})$, using again the additional scheme estimate (1).

Theorem 4.3 *Assuming $|\delta_t \mathbf{e}_d^1| \le C\, h$, then*

$$\|\mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{H}^1)} \le C\, h \quad \text{ and } \quad \|e_{p,d}^{m+1}\|_{l^\infty(L^2)} \le C\, h.$$

Outline of the proof: It is based on the error estimate

$$\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\, h,$$

which is obtained as in step 1 of the proof of Theorem 4.1.

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