Blow up, Bessel functions and a maximal square estimate for the Schödinger equation

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The solution to the linear Schrödinger equation in $\mathbb{R}^{n+1}_+$, $\partial_t u(x,t) = \frac{\mathrm{i}}{2\pi}\Delta_x u(x,t)$, with initial datum f, is given formally by

$$S_t f(x) = u(x,\, t) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \mathrm{e}^{-2\pi \mathrm{i}|\xi|^2 t} \mathrm{e}^{2\pi \mathrm{i} \xi\cdot x}\, \mathrm{d}\xi.$$

There is a fundamental question in this setting and is that of determining the minimal smoothness on the initial value function f, needed for the almost everywhere convergence

$$\lim_{t\to 0^+} u(x,\, t) = f(x), \qquad \text{a.e.} \tag{1}$$

Lennart Carleson proved that, for $n = 1$, if f belongs to the Sobolev class $H^{1/4}(\mathbb{R})$ then (1) holds. Moreover, the exponent $1/4$ is best possible. It is conjectured that this condition suffices in any dimension.

In this work we show that the conjecture is true on quadratic means, that is, for every $f \in H^{1/4}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ we have

$$\lim_{t\to 0^+} \int_{S^{n-1}} |S_t f(x_0 + r\omega) - f(x_0 + r\omega)|^2\, \mathrm{d}\sigma(\omega) = 0, \quad \text{a.e. } r.$$

This is obtained via the boundedness of certain maximal square function associated to the natural projection on spaces of spherical harmonics.

In the process we study blow up regions and exponents β for the estimates

$$\int_I \mathrm{e}^{\mathrm{i}as^2} J_\nu(s) \frac{\mathrm{d}s}{s^\beta} = \mathcal{O}(1),$$

to hold independently of $\nu \in \mathbb{N}/2$, the interval I and $a \in \mathbb{R}$. Here J_ν denotes the Bessel function of order ν.