III Curso Internacional de Análisis Armónico en Andalucía

Spaces of smooth functions

by

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Abstract: We will pose several situations in analysis where some classes of smooth functions play a fundamental role. In connection with the study of Laplace equation, we shall analyze the behaviour of the fractional integral operator on L^p spaces, where BMO and Lipschitz spaces arise in a natural way. As a generalization, we will present and study a family of spaces introduced by Spanne. In particular we will be interested in identifying those members of the family containing only continuous functions. Finally we shall present a brief description of Besov spaces and their connection with a problem of non-linear approximation of a function by its wavelet expansion.

La Rábida, September 2007

1 Fractional integration

The fractional integral operator arises in a natural way when solving problems involving differential operators. From elementary one variable calculus we know that integration and differentiation are inverse operations. This is basically the content of the Fundamental Theorem of Calculus. The picture is not that simple in higher dimensions where the most interesting situations occur. In order to solve partial differential equations, even in a theoretical framework, we must deal with operators involving inverses of "derivatives". Fractional integrals are in many cases the key operators to handle such inverses. The basic identity that leads to a generalization of the fundamental theorem of calculus in one variable, i.e., $\int_a^t f'(s)\,ds = f(t) - f(a)$, is the following

$$f(x) = c_n \int_{\mathbb{R}^n} \frac{\langle \nabla f(y), x-y \rangle}{|x-y|^n}\, dy,$$

where f denotes a function defined on $\mathbb{R}^n$, with compact support and continuous partial derivatives. In fact, let $B(x,R)$ be a ball centered at x and with radius large enough to contain the support of f. For each unit direction y' we may apply the one dimensional result to get

$$f(x) = \int_0^R D_{y'} f(x - ty')\, dt = \int_0^\infty \langle \nabla f(x-ty'), y' \rangle\, dt.$$

Integrating both sides over all the directions y' we obtain

$$\begin{aligned} f(x) &= c_n \int_{S^{n-1}} \int_0^\infty \frac{\langle \nabla f(x-ty'), ty' \rangle}{t^n}\, t^{n-1}\, dt\, dy' \\ &= c_n \int_{\mathbb{R}^n} \frac{\langle \nabla f(x-y), y \rangle}{|y|^n}\, dy = c_n \int_{\mathbb{R}^n} \frac{\langle \nabla f(y), x-y \rangle}{|x-y|^n}\, dy. \end{aligned}$$

From here it follows immediately that

$$|f(x)| \le c_n \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy.$$

Now we introduce the definition of the Fractional Integral Operator of order α, $0 < \alpha < n$, by the expression

$$I_\alpha g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, dy.$$

It follows that, taking $\alpha = 1$ (as long as n is greater than one),

$$|f(x)| \le c_n I_1(|\nabla f|).$$

As a consequence we may say that an improvement on the integrability of the function $I_\alpha(g)$ with respect to that of g, i.e., some boundedness results of I_α on Lebesgue spaces, would lead to obtain a better degree of integrability for a function f from assumptions on the size of its gradient. As an example, if we start with a function in L^2 whose gradient belongs also to L^2 and we are able to prove that the Fractional Integral operator for $\alpha = 1$ maps L^2 into L^q for some $q > 2$, we might conclude that f has in fact a better local integrability than that originally assumed. This type of result is known as one of the "immersion Sobolev's theorems" and it turns to be a fundamental tool in proving regularity properties for weak solutions to some uniformly elliptic partial differential equations, like the Laplace equation. In a similar way, results on the behavior of I_α over smooth function spaces are fundamental for obtaining regularity properties for classical solutions of such kind of equations. During the last fifty years, Fractional Integral operators have been intensively studied, not only in the present context but in more general situations to englobe larger classes of equations.

Another way of looking at the relationship between Fractional Integral operators and derivatives is by studying their Fourier transforms. Since they are convolution operators, it is enough to know the Fourier transform of the kernel $k(x) = |z|^{\alpha-n}$. A homogeneity argument allows us to see that $\hat{k}(\xi)$ is, up to a constant, $|\xi|^{-\alpha}$. On the other hand, if we compute the Fourier transform of $(-\Delta)^{\alpha/2}$ using distributional calculus, we easily find that it is a constant times $|\xi|^\alpha$. Therefore the composition of I_α with $(-\Delta)^{\alpha/2}$, whenever possible, gives the identity.

We shall start our study by stating some classical results concerning the behavior of these operators on the Lebesgue space $L^p(\mathbb{R}^n)$, that is, the set of measurable functions defined on $\mathbb{R}^n$ such that $|f|^p$ is integrable.

Theorem 1 (Hardy-Littlewood-Sobolev). *Let $0 < \alpha < n$ and $1 < p < n/\alpha$. Then I_α is a bounded operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with $1/q = 1/p - \alpha/n$, that is, there exists a constant C such that*

$$\|I_\alpha f\|_q \leq C\|f\|_p.$$

Remark. It is a pleasant exercise to check that, because of the homogeneity of the kernel, if the operator I_α maps L^p into L^q, the relationship $1/q = 1/p - \alpha/n$ must hold. In fact, choosing g with $\|g\|_{L^p} = 1$, the above norm inequality applied to $f(x) = g(\lambda x)$ gives that

$$\lambda^{-\alpha-n/q} \leq C\lambda^{-n/p},$$

should be true for any $\lambda > 0$. That is possible only if $\alpha + n/q = n/p$, which is the same as $1/q = 1/p - \alpha/n$. ◊

In order to prove the theorem we first introduce a new space, a little bit larger than L^q, named weak-L^q or $L^{q,*}$ for short. Given a measurable function f, let us denote by μ_f its distribution function, that is, for $\lambda > 0$

$$\mu_f(\lambda) = |\{x : |f(x)| > \lambda\}|.$$

We will say $f \in L^{q,*}(\mathbb{R}^n)$ if there is a constant c such that

$$\mu_f(\lambda) \leq \frac{c}{\lambda^q} \quad \text{for all } \lambda > 0.$$

The infimum of such constants raised to $1/p$-th power turns to be a norm in this space as long as $1 \leq q < \infty$, and moreover it is complete. The well known Tchebycheff's inequality

$$\mu_f(\lambda) \leq \frac{1}{\lambda^q} \int_{\mathbb{R}^n} |f|^q,$$

implies that $L^q \subset L^{q,*}$, continuously. On the other hand, it is straightforward to check that $g(x) = 1/|x|^{n/q}$ belongs to $L^{q,*}$ but, however, g does not belong to L^q.

Now, if a given operator T is bounded from L^p into $L^{q,*}$ we shall say that it is of weak type (p, q), while we shall say that T is of strong type (p, q) whenever it is bounded from L^p into L^q. From the above remark we deduce that any strong type operator is of weak type. However the converse might be not true, as we shall illustrate later. We shall make use of a famous theorem due to Marcinkiewicz that will allow us to derive strong boundedness results from weak type inequalities. We give the precise statement (for a proof see [St]).

Theorem 2 (Marcinkiewicz's interpolation theorem)**.** *Let p_0, p_1, q_0, q_1 be real numbers such that $1 \leq p_i \leq q_i \leq \infty$, $p_0 < p_1$ y $q_0 \neq q_1$. Let T be a sublinear operator which is simultaneously of weak type (p_0, q_0) and (p_1, q_1). Then for each θ, $0 < \theta < 1$, with $1/p = (1-\theta)\,1/p_0 + \theta\,1/p_1$ and $1/q = (1-\theta)\,1/q_0 + \theta\,1/q_1$, we have that T is of strong type (p, q), that is,*

$$\|Tf\|_q \leq A\|f\|_p.$$

(When $q_i = \infty$ weak type means $\|Tf\|_{q_i} \leq A_i\|f\|_{p_i}$.)

We shall also use the the very well known Young's inequality for convolutions, namely

$$\|f * g\|_r \leq \|f\|_s \|g\|_t,$$

where $1 \leq s, t \leq \infty$ y $1 + 1/r = 1/s + 1/t$.

Proof of theorem 1. We will prove that for $1 \leq p < n/\alpha$, $1/q = 1/p - \alpha/n$, the operator I_α satisfies

$$\big|\{x : |I_\alpha f|(x) > \lambda\}\big| \leq \frac{c}{\lambda^q} \Big(\int |f|^p\Big)^{q/p}.$$

In other words, I_α is of weak type $(p,q), 1 \leq p < n/\alpha$. From here, by means of Marcinkiewicz's interpolation theorem, we will obtain the strong type (p,q), in the range $1 < p < n/\alpha$.

For each $\eta > 0$ we split the kernel $K(x) = |x|^{\alpha-n}$ in

$$K = K_0 + K_\infty,$$

where $K_0 = K\chi_{B(0,\eta)}$ and $K_\infty = K\chi_{B^c(0,\eta)}$.

If f belongs to L^p, $K_0 * f$ as well as $K_\infty * f$ are finite a.e.. This is so since K_0 is an L^1 function while K_∞ is in $L^{p'}$, and then an application of Young's inequality gives that $K_0 * f$ belongs to L^p, and that $K_\infty * f$ is in L^∞ and, consequently, finite almost everywhere. Moreover, straightforward calculations show that

$$\|K_0\|_1 \leq c_0\eta^\alpha, \quad \|K_\infty\|_{p'} = c_1\eta^{-n/q}.$$

Now, let us observe that

$$\begin{aligned}|\{x : |K * f|(x) > 2\lambda\}| \\ &\leq \big|\{x : |K_0 * f|(x) > \lambda\}\big| + \big|\{x : |K_\infty * f|(x) > \lambda\}\big| \\ &= I + II.\end{aligned}$$

To estimate I we use Tchebycheff's and Young's inequalities to get

$$I \leq \frac{1}{\lambda^p}\,\|K_0 * f\|_p^p \leq \frac{1}{\lambda^p}\,\|K_0\|_1^p\,\|f\|_p^p \leq c_0^p\Big(\frac{\eta^\alpha\|f\|_p}{\lambda}\Big)^p.$$

On the other hand, since for almost every x,

$$|K_\infty * f|(x) \leq \|K_\infty\|_{p'}\,\|f\|_p \leq c_1\eta^{-n/q}\|f\|_p,$$

choosing η such that $c_1\eta^{-n/q}\|f\|_p = \lambda$ we obtain that $II = 0$.

Consequently, for this value of η we have

$$\big|\{x : |K * f|(x) > 2\lambda\}\big| \leq \frac{c_0^p}{c_1^{pq\alpha/n}}\Big(\frac{\|f\|^{q\alpha/n}}{\lambda^{q\alpha/n}}\,\frac{\|f\|_p}{\lambda}\Big)^p = c\Big(\frac{\|f\|_p}{\lambda}\Big)^q,$$

since from the relationship between p and q it follows that $1+q\alpha/n = q/p$. □

Therefore we have shown that I_α is of weak type (p,q) for p in the interval $[1, n/\alpha)$. Since any p in the open interval $(1, n/\alpha)$ may be seen as an intermediate point between two values p_0 and p_1 belonging to the same interval, we may conclude via interpolation that I_α is of strong type for $p \in (1, n/\alpha)$ and q such that $1/q = 1/p - \alpha/n$.

Remark. In the above proof we have seen that I_α is also of weak type (p,q) when $p = 1$ and $q = n/(n-\alpha)$.

Moreover it can be shown that it is not of strong type in the extreme point. In fact, if we take a sequence of functions f_k, $\|f_k\|_1 = 1$ tending to the Dirac delta we will have

$$I_\alpha f_k(x) = K * f_k \to c_n/|x|^{n-\alpha},$$

for almost every $x \in \mathbb{R}^n$. Therefore, if the strong type inequality were true we would have

$$\|K * f_k\|_{n/(n-\alpha)} \leq A,$$

and by Fatou's Theorem, we would arrive to

$$\int_{\mathbb{R}^n} |x|^{-n}\, dx < \infty,$$

which is obviously false. ♢

We state our observation as another boundedness result for I_α.

Theorem 3. *Let $0 < \alpha < n$. The operator I_α is of weak type $(1, n/(n-\alpha))$ but not of strong type.*

It is also not difficult to check that in the other end point $p = n/\alpha$, I_α is not of strong type $(n/\alpha, \infty)$ as may be expected. In this case it is enough to take $f(x) = |x|^{-\alpha}(\log 1/|x|)^{-r\alpha/n}\chi_{B(0,1/2)}(x)$, with $1 < r \leq n/\alpha$, which belongs to $L^{n/\alpha}$ and observe that

$$I_\alpha f(x) = \int_{|y|\leq 1/2} \frac{|y|^{-\alpha}}{|x-y|^{n-\alpha}} (\log 1/|y|)^{-r\alpha/n}\, dy,$$

is a continuous function for $x \neq 0$, and also

$$\lim_{x\to 0} I_\alpha f(x) = \int_{|y|\leq 1/2} (\log 1/|y|)^{-r\alpha/n}\, |y|^{-n}\, dy = \infty,$$

since $1 - r\alpha/n \geq 0$, giving that $I_\alpha f$ is not essentially bounded. Then, a natural question arises. What can be said about $I_\alpha(f)$ for a function $f \in$

$L^{n/\alpha}$? Certainly we should enlarge the space L^∞ so as to allow functions behaving locally as the logarithm at the origin. The appropriate space is known as *BMO* (bounded mean oscillation) or the John-Nirenberg space (see [JN]) and it is defined as:

$$BMO = \Big\{ f \in L^1_{loc} : \|f\|_* = \sup_B \frac{1}{|B|} \int_B |f(x) - m_B f|\, dx < \infty \Big\},$$

where the supremum is taken over the family of balls in $\mathbb{R}^n$, and $m_B f$ denotes the average of f over the ball B, that is, $m_B f = \frac{1}{|B|} \int_B f$.

If we want $\| \ \|_*$ to be a norm, we must identify those functions whose difference is a constant.

With this notation we will be able to prove the following result:

Theorem 4. *Let $f \in L^{n/\alpha}$ and having compact support. Then, $I_\alpha f$ is finite almost everywhere and*

$$\|I_\alpha f\|_* \le C \|f\|_{n/\alpha}.$$

Proof. Since such f belongs for instance to L^p, $1 < p < n/\alpha$, then $I_\alpha f \in L^q$, $1/q = 1/p - \alpha/n$, and hence locally integrable.

Let $B = B(x_0, r)$ be a ball. We decompose $f = f_1 + f_2$ with $f_1 = f\chi_{\tilde{B}}$ where $\tilde{B} = B(x_0, 2r)$.

Now,

$$\frac{1}{|B|} \int_B |I_\alpha f - m_B I_\alpha f| \le \frac{2}{|B|} \int_B |I_\alpha f_1| + \frac{1}{|B|} \int_B |I_\alpha f_2 - m_B I_\alpha f_2| = I + II.$$

But if we choose p and q such that $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, we obtain

$$\frac{1}{|B|} \int_B |I_\alpha f_1| \le \big(\frac{1}{|B|} \int_B |I_\alpha f_1|^q \big)^{1/q} \le C \frac{1}{|B|^{1/q}} \big(\int |f_1|^p \big)^{1/p},$$

in view of Theorem 1. Applying Hölder's inequality with $r = n/\alpha p > 1$ and $r' = n/(n - \alpha p)$ we get

$$I \le c \, \frac{|\tilde{B}|^{(n-\alpha p)/np}}{|B|^{1/q}} \, \|f\|_{n/\alpha} = c\, \|f\|_{n/\alpha}.$$

On the other hand

$$II \le \frac{1}{|B|^2} \int_B \int_B \int_{\tilde{B}^c} |f_2(y)| \, \big| |x - y|^{\alpha - n} - |z - y|^{\alpha - n} \big| \, dy\, dz\, dx.$$

Since $x, z \in B$ and $y \in \tilde{B}^c$, $|x-y| \geq r$, $|z-y| \geq r$. An application of the mean value theorem leads to

$$\big||x-y|^{\alpha-n} - |z-y|^{\alpha-n}\big| \leq c\,|x-z|\,\theta^{\alpha-n-1},$$

being θ an intermediate value between $|x-y|$ and $|z-y|$. Since in our situation both values are equivalent to $|x_0-y|$, the last expression is bounded by $cr|x_0-y|^{\alpha-n-1}$, and then

$$\begin{aligned} II \leq cr \int_{|x_0-y|>2r} |f(y)|\,|x_0-y|^{\alpha-n-1}\,dy \\ \leq cr\|f\|_{n/\alpha}\Big(\int_{|x_0-y|>r} |x_0-y|^{(\alpha-n-1)\,n/(n-\alpha)}\,dy\Big)^{(n-\alpha)/n}. \end{aligned}$$

Changing to polar coordinates the last integral equals to a constant times

$$\int_r^\infty \rho^{-n-n/(n-\alpha)}\rho^{n-1}\,d\rho = c\,r^{-n/(n-\alpha)},$$

and therefore we also obtain

$$II \leq c\,\|f\|_{n/\alpha}. \qquad \square$$

Remark. We have stated the theorem only for $I_\alpha f$ with f in $L^{n/\alpha}$ and having compact support. Let us notice that for such functions, if we define

$$\begin{aligned} \tilde{I}_\alpha f(x) &= \int_{\mathbb{R}^n}\Big(\frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{B^c(0,1)}}{|y|^{n-\alpha}}\Big) f(y)\,dy \\ &= I_\alpha f(x) - \int_{|y|\geq 1} \frac{f(y)}{|y|^{n-\alpha}}\,dy = I_\alpha f(x) - C, \end{aligned}$$

we would obtain that $\tilde{I}_\alpha f$ and $I_\alpha f$ are the same as functions in BMO. On the other hand, it is easy to see that for $f \in L^{n/\alpha}$, $\tilde{I}_\alpha f$ is finite almost everywhere and moreover locally integrable. In fact, let $B_R = B(0,R)$ with $R > 1$ and $x \in B_R$. We write

$$\begin{aligned} \tilde{I}_\alpha f(x) = \int_{|y|\leq 2R} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy + \int_{1\leq|y|\leq 2R} \frac{f(y)}{|y|^{n-\alpha}}\,dy \\ + \int_{|y|\geq 2R}\Big[\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}}\Big] f(y)\,dy. \end{aligned}$$

The first term in the sum gives a function in L^1_{loc} since it is the fractional integral of a $L^{n/\alpha}$ function with compact support. The second integral is a finite quantity and independent of x since $1/|y|^{n-\alpha} \leq 1$ and, being f in $L^{n/\alpha}$, is locally integrable. Finally, for x in B_R, the quantity between brackets is a difference of two values of the function $t^{\alpha-n}$ away from the origin, and hence the mean value theorem may be applied to bound the integrand by $C|x|/|y|^{n-\alpha+1}$, since again $|x-y| \simeq |y|$. Clearly, this last function belongs to $L^{n/(n-\alpha)}$, and then Hölder's inequality gives that the third integral is bounded by $C|x|$ which is integrable on B_R.

From these observations we can say that $\tilde{I}_\alpha f$ provides an extension of $I_\alpha f$ for general functions belonging to $L^{n/\alpha}$ and not necessarily with compact support. ♢

Similar considerations hold for $n/\alpha \leq p < n/(\alpha-1)^+$. It turns out that $\tilde{I}_\alpha f$ is also well defined for $f \in L^p$, giving a class of locally integrable functions that differ by a constant. In fact the same argument applies and all we need is that $1/|y|^{n-\alpha} \in L^{p'}_{loc}$ and $1/|y|^{n-\alpha+1}\chi_{B^c_R} \in L^{p'}$, and clearly both are true in the stated range.

A new question therefore arises: what can be said about the image of L^p under $\tilde{I}_\alpha$ when $n/\alpha < p < n/(\alpha-1)^+$?

From the above remark we know that $\tilde{I}_\alpha f$ is locally integrable and the proof of theorem 3 can be followed step by step; the only difference is that when estimating the averages in terms of $\|f\|_p$ instead of $\|f\|_{n/\alpha}$, we will obtain $C\,\|f\|_p\,|B|^{\alpha/n-1/p}$ on the right hand side.

In this way we would get an estimate of the type

$$\frac{1}{|B|^{\alpha/n-1/p}}\,\frac{1}{|B|}\int_B |\tilde{I}_\alpha f - m_B \tilde{I}_\alpha f| \leq C\|f\|_p,$$

for p such that $n/\alpha < p < n/(\alpha-1)^+$. Let us observe that in such situation the exponent $\alpha/n - 1/p$ is always positive and less than $1/n$. Moreover, the above inequality for $p = n/\alpha$ gives the statement of Theorem 3.

Then, for a given $0 \leq \beta < 1$ we introduce the space

$$BMO_\beta = \Big\{ f \in L^1_{loc} : \sup_B \frac{1}{|B|^{\beta/n}}\,\frac{1}{|B|}\int_B |f - m_B f| < \infty \Big\}.$$

When $\beta = 0$ we recover BMO and for $\beta > 0$, as we shall see in the next section, it coincides with a very well known space of smooth functions.

2 Functions with controlled mean oscillation

As a generalization of the spaces we just introduced, when trying to describe the image of L^p ($p > n/\alpha$) under the fractional integration, S. Spanne [Sp] defined the BMO_φ spaces as the class of functions whose mean oscillation is controlled by φ, a fixed non-decreasing and positive function defined on $(0,\infty)$. More precisely,

$$BMO_\varphi = \Big\{ f \in L^1_{loc} : \sup_B \frac{1}{\varphi(|B|^{1/n})} \frac{1}{|B|} \int_B |f - m_B f| < \infty \Big\},$$

and moreover, if we denote by $\| \; \|_{*,\varphi}$ that supremum, taking over all the balls in $\mathbb{R}^n$, the space BMO_φ, turns to be a Banach space, after identifying those functions that differ by a constant.

Clearly for $\varphi(t) = t^\beta$, $0 \le \beta < 1$, we have the spaces introduced in the previous section. In particular, for $\beta = 0$ we recover the John-Nirenberg space. These spaces were firstly studied by Campanato [C] and Meyers [M] in connection with the study of regularity of solutions of elliptic partial differential equations.

In this section we plan to study some properties of these spaces. In particular, it is obvious that BMO ($\beta = 0$) contains non continuous functions (obviously $L^\infty \subset BMO$), while in [C] and [M] it is shown that for $0 < \beta < 1$, all the functions are continuous and, moreover, their modulus of continuity is not worse than t^β.

Spanne [Sp], considered the problem of smoothness for functions in BMO_φ, posing the questions of finding conditions on φ to guarantee that BMO_φ contains only smooth functions and when such situation does not occur.

To answer these questions we introduce the space of Lipschitz-φ functions, as those functions whose modulus of continuity is controlled by φ, i.e.

$$\Lambda_\varphi = \Big\{ f : \omega_f(t) = \sup_{|x-y|\le t} |f(x) - f(y)| \le c\,\varphi(t) \Big\}.$$

It is immediate to check that $\Lambda_\varphi \subset BMO_\varphi$ and also that $\Lambda_\varphi = L^\infty/c$ when $\varphi(t) \simeq 1$ (Here L^∞/c means that we have identified functions differing a.e. by a constant.)

In the next theorem we state the results by Spanne.

Theorem 5. *Let φ be a non-decreasing and positive function. Then we have*

(a) If the function φ also satisfies $\int_0^\delta \varphi(t)\,dt/t < \infty$ for some $\delta > 0$ then any function in BMO_φ is continuous and moreover $\omega_f(s) \le c \int_0^s \varphi(t)\,dt/t$.

(b) If $\varphi(t)/t$ is non increasing and $\int_0^\delta \varphi(t)\,dt/t$ diverges, then the space BMO_φ contains discontinuous and locally unbounded functions.

Corollary 6. *If φ is such that $\int_0^s \varphi(t)\,dt/t < \infty$ for some $\delta > 0$, denoting by $\tilde\varphi(s) = \int_0^s \varphi(t)\,dt/t$, it follows that $BMO_\varphi \subset \Lambda_{\tilde\varphi}$.*

We will not show the result (a) of Spanne in its full generality. Instead, to make the computations easier, we are going to assume that $\varphi(t)/t$ is non increasing also for the proof of (a).

We shall make use of the following simple lemma.

Lemma 7. *Let $f \in BMO_\varphi$ and $B \subset \overline{B}$ two balls in $\mathbb{R}^n$. Then*

$$|m_B f - m_{\overline{B}} f| \leq \|f\|_{*,\varphi} \frac{|\overline{B}|}{|B|} \varphi(|\overline{B}|^{1/n}).$$

Proof.

$$\begin{aligned} |m_B f - m_{\overline{B}} f| &= \frac{1}{|B|} \int_B |f - m_{\overline{B}} f| \\ &\leq \frac{|\overline{B}|}{|B|} \frac{1}{|\overline{B}|} \int_{\overline{B}} |f - m_{\overline{B}} f| \leq \frac{|\overline{B}|}{|B|} \varphi(|\overline{B}|^{1/n}) \|f\|_{*,\varphi}. \quad \square \end{aligned}$$

Proof of theorem 5 (a). Let us start by noticing that $\varphi(t)/t$ non increasing implies that for any fixed $a \geq 1$, there is a constant c such that $\varphi(at) \leq c\,\varphi(t)$. On the other hand if $a < 1$ such inequality holds with constant one, since φ is non-decreasing. It is also clear that $\varphi(t/2) \leq c \int_{t/2}^t \varphi(s)\, ds/s \leq c \int_0^t \varphi(s)\, ds/s$. Hence $\varphi(t) \leq c\,\tilde{\varphi}(t)$.

Let $x, y \in \mathbb{R}^n$ and $B = B(x, |x-y|)$, $B' = B(y, |x-y|)$ and $\tilde{B} = B(x, 2|x-y|)$.

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - m_B f| + |f(y) - m_{B'} f| \\ &\quad + |m_{B'} f - m_{\tilde{B}} f| + |m_{\tilde{B}} f - m_B f| = I + II + III + IV. \end{aligned}$$

Since both, B y B' are contained in $\tilde{B}$, the terms III y IV, according to lemma 7, are bounded by

$$2^n \varphi(|\tilde{B}|^{1/n}) \|f\|_{*,\varphi} \leq c\, 2^n \varphi(|x-y|) \|f\|_{*,\varphi} \leq c \|f\|_{*,\varphi} \int_0^{|x-y|} \varphi(t) \frac{dt}{t}.$$

The terms I y II are quite similar, so we only bound the first. We set $B_i = B(x, 2^{-i}|x-y|)$ for $i \geq 1$ y $B_0 = B$. Then we have

$$|f(x) - m_B f| \leq |f(x) - m_{B_m} f| + \sum_{i=0}^{m-1} |m_{B_{i+1}} f - m_{B_i} f|.$$

Since f is locally integrable, Lebesgue's differentiation theorem applies. Let us assume that x is in fact a Lebesgue point. Then, taking limit for

$m \to \infty$, the first term on the right hand side goes to zero, and applying lemma 7 to each term in the series we get

$$|f(x) - m_B f| \leq \sum_{i=0}^{\infty} |m_{B_{i+1}} f - m_{B_i} f| \leq c \, \|f\|_{*,\varphi} \sum_{i=0}^{\infty} \varphi(2^{-i}|B|^{1/n})$$
$$\leq C' \, \|f\|_{*,\varphi} \sum_{i=0}^{\infty} \int_{2^{-i}}^{2^{-i+1}} \varphi(t|B|^{1/n}) \, \frac{dt}{t} \leq C \, \|f\|_{*,\varphi} \int_0^1 \varphi(t|B|^{1/n}) \, \frac{dt}{t}.$$

Since $|B|^{1/n} = \omega_n \, |x - y|$ with $\omega_n = |B(0,1)|^{1/n}$, performing the change of variables $s = t|x - y|$ and using that $\varphi(as) \leq c \, \varphi(s)$ it follows that

$$I \leq c \, \|f\|_{*,\varphi} \int_0^{|x-y|} \varphi(s) \, \frac{ds}{s},$$

for some appropriate constant c. Therefore, part (a) of the theorem is proved under the extra assumption $\varphi(t)/t$ non-increasing. □

Before proceeding with the proof of part (b), let us observe that a function f satisfying the property: for any ball B there is a constant C_B such that

$$\frac{1}{|B|} \int_B |f - C_B| \leq A \, \varphi(|B|^{1/n}),$$

with A independent of the ball B, certainly belongs to BMO_φ, and moreover $||f||_{*,\varphi} \leq 2A$. In fact,

$$\frac{1}{|B|} \int_B |f - m_B f| \leq \frac{1}{|B|} \int_B |f - C_B| + |C_B - m_B f|$$
$$\leq A\varphi(|B|^{1/n}) + \frac{1}{|B|} \int_B |f - C_B| \leq 2A\varphi(|B|^{1/n}).$$

Consequently, in order to prove that a function does belong to BMO_φ we may use any constant C_B instead of $m_B f$.

Proof of theorem 5 (b). We set

$$h(x) = \int_{|x|}^{1} \frac{\varphi(t)}{t} \, dt.$$

Then h is continuous at $x \neq 0$ and under the assumptions on φ, it is discontinuous at $x = 0$ and unbounded nearby. To check that $h \in BMO_\varphi$, it is enough to consider balls $B(z, r)$ with $z \neq 0$.

We set $B = B(z,r)$, $z_B = z + r\frac{z}{|z|}$ and $C_B = h(z_B)$. Let us notice that $|z_B| = |z| + r$ and that for $x \in B$, $|x| \le |x - z| + |z| \le |z| + r$. Hence for $x \in B$,

$$|h(x) - C_B| = |h(x) - h(z_B)| = \int_{|x|}^{|z|+r} \varphi(t)\,\frac{dt}{t}.$$

In order to estimate the oscillation, let us consider first the case $|z| < 2r$. In this situation we have

$$\begin{aligned}\int_B |h(x) - C_B| &= \int_B \int_{|x|}^{|z|+r} \varphi(t)\,\frac{dt}{t}\,dx \le \int_0^{|z|+r} \frac{\varphi(t)}{t}\Big(\int_{|x|\le t} dx\Big)\,dt\\ &\le C\,\varphi(|z|+r)\int_0^{|z|+r} t^{n-1} \le C\,\varphi(3r)\,r^n \le C\,\varphi(|B|^{1/n})|B|,\end{aligned}$$

where we have used φ non-decreasing, $|z| < 2r$ and $\varphi(ar) \le C\varphi(r)$.

Now if $|z| > 2r$, the distance from the origin to the ball is at least r. In fact, if $x \in B$, $|x| \ge |z| - |z - x| \ge |z| - r \ge r$.

In this way

$$\int_B |h(x) - C_B| \le \int_B \Big(\int_r^{|z|+r} \varphi(t)\,\frac{dt}{t}\Big)\,dx \le |B|\,\frac{\varphi(r)}{r}\,2r = 2|B|\,\varphi(|B|^{1/n}),$$

where we have used that $\varphi(t)/t$ is non-increasing. □

Remarks.

1. It is worth noting that the proof of $h \in BMO_\varphi$ does not make use of the divergence of the integral, we just used φ non-decreasing and $\varphi(t)/t$ non increasing.

2. For $\varphi(t)/t$ non increasing, (a) and (b) imply that if $\int_0^\delta \varphi(t)\,dt/t$ diverges then $\Lambda_\varphi \subsetneq BMO_\varphi$. In fact, when $\varphi(0^+) = 0$, all the functions in Λ_φ are continuous and when $\varphi(0^+) > 0$ they are bounded (locally). On the other hand, if the integral converges and $\varphi \simeq \tilde{\varphi}$, then $\Lambda_\varphi = BMO_\varphi$. Conversely, it can be seen that if both spaces agree, not only the integral must converge (a consequence of (b)) but $\varphi \simeq \tilde{\varphi}$ must hold. Indeed, by the previous remark, $h \in BMO_\varphi$ and hence $h \in \Lambda_\varphi$. Therefore,

$$|h(x) - h(0)| \le C\varphi(|x|).$$

But, according to the definition of h,

$$|h(x) - h(0)| = \int_0^{|x|} \varphi(t)\,\frac{dt}{t} = \tilde{\varphi}(|x|).$$

Then $\tilde{\varphi}(r) \le C\varphi(r)$, for any positive r. Since the converse inequality always holds, we arrive to $\varphi \simeq \tilde{\varphi}$.

3. An example where the assumptions made in (b) hold is $\varphi(t) \equiv 1$. In such case the function h is

$$h(|x|) = \log(1/|x|),$$

which is the classical example of unbounded function (even locally) which does belong to BMO. $\diamondsuit$

3 Smooth function spaces and wavelets

Besides BMO_φ, there are other families of spaces that generalize Lipschitz-α spaces. We will introduce another line of spaces and we shall present a problem arising in non-linear approximation where they become the appropriate spaces. We will follow closely the exposition given in the book by Wojtaszczyk [W, Ch. 9].

In the sequel, for simplicity, we will restrict our functions to one dimension, even though most of the results have an extension to higher dimensions.

As we have seen, a Lipschitz function is defined in terms of its pointwise modulus of continuity, i.e.,

$$\omega_f(t) = \sup_{|h|\le t} \sup_x |f(x+h) - f(x)| = \sup_{|h|\le t} \|f(x+h) - f(x)\|_\infty,$$

which measures in some sense, the size of the difference between a function and its translation. Since there are many ways of measuring the size of a function, it is natural to introduce the p-modulus of continuity by

$$\omega_p(f,t) = \sup_{|h|\le t} \|f(x+h) - f(x)\|_p.$$

Clearly, $\omega_\infty(f,t) = \omega_f(t)$.

Next we establish several simple properties of ω_p:

(i) $\omega_p(f,t)$ is a non-decreasing function of t.

(ii) If $1 \le p < \infty$ and $f \in L^p$, then $\lim_{t\to 0} \omega_p(f,t) = 0$, and moreover $\omega_p(f,t) \le 2\|f\|_p$ for $t > 0$.

(iii) $\omega_p(f,mt) \le m\,\omega_p(f,t)$ if $m \in \mathbb{N}$.

(iv) $\lim_{t\to 0} \frac{1}{t}\,\omega_p(f,t) = 0 \;\Rightarrow\; f = \text{constant}$.

Clearly (i) and (iii) hold. For (ii) the claim on the limit is obvious for smooth functions with compact support, and the result follows by the density of such functions in L^p. Finally from (iii) we get

$$\omega_p(f,t) = \omega_p(f, mt/m) \leq \frac{\omega_p(f, t/m)}{t/m}\, t.$$

Making m tend to infinity, the assumption in (iv) implies that $\omega_p(f,t) = 0$ for each $t > 0$ and then f equal a.e to a constant.

A function belongs to a Lipschitz-α space whenever $\sup_{t>0} t^{-\alpha}\, \omega_\infty(f,t) < \infty$. Again, we may change the sup-norm by a different norm, to introduce a new family of spaces: the non homogeneous Besov spaces $\dot{B}^p_{\alpha,s}$, with $0 < \alpha \leq 1$ and $1 \leq p, s \leq \infty$, as the set of functions such that $\|f\|_{p,\alpha,s} < \infty$, where

$$\|f\|_{p,\alpha,s} = \begin{cases} \left(\int_0^\infty \left[t^{-\alpha} \omega_p(f,t) \right]^s \frac{dt}{t} \right)^{1/s} & \text{if } 1 \leq s < \infty, \\ \sup_{t>0}\, t^{-\alpha}\, \omega_p(f,t) & \text{if } s = \infty. \end{cases}$$

In fact, $\|\ \|_{p,\alpha,s}$ are seminorms and they vanish on constant functions. If we want to work with a norm we should identify functions differing by a constant, but the resulting spaces may not be complete. When these spaces are completed, they involve not only functions but also distributions, and their treatment becomes more difficult.

One way to bypass this difficulty is to introduce the so called homogeneous Besov spaces $B^p_{\alpha,s}$, as the set of functions in L^p such that $\|f\|_{p,\alpha,s} < \infty$. In this way, it turns to be a Banach space with respect to the norm $\|f\|_p + \|f\|_{p,\alpha,s}$.

The seminorm $\|\ \|_{p,\alpha,s}$ has a discrete version as it is easy to check.

Proposition 8. *For p, α, s as above, there exist positive constants c and C such that*

$$c\, \|f\|_{p,\alpha,s} \leq \sum_{j \in \mathbb{Z}} 2^{\alpha j s}\, \omega_p(f, 2^{-j})^s \leq C\, \|f\|_{p,\alpha,s}.$$

Proof. Splitting the integral into dyadic intervals and using (i) and (iii) we get

$$\int_0^\infty \left[t^{-\alpha} \omega_p(f,t) \right]^s \frac{dt}{t} = \sum_{j \in \mathbb{Z}} \int_{2^{-j}}^{2^{-j+1}} \omega_p(f,t)^s \frac{dt}{t^{\alpha s + 1}} \leq 2^s \sum_{j \in \mathbb{Z}} 2^{\alpha j s}\, \omega_p(f, 2^{-j})^s.$$

Also by (i) the integral is bounded below by

$$2^{-\alpha s - 1} \sum_{j \in \mathbb{Z}} 2^{\alpha j s}\, \omega_p(f, 2^{-j})^s,$$

and the proposition is proved for $s < \infty$. The case $s = \infty$ follows similarly, just replacing integrals and sums by suprema. □

To illustrate a situation where Besov's spaces appear in a natural way, we shall introduce, in an informal way, some basics concepts and facts from the one dimensional wavelet theory.

A *wavelet* on $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that the family of functions

$$\psi_{jk}(t) = 2^{j/2}\,\psi(2^j\,t - k) \quad \text{with } j, k \in \mathbb{Z},$$

gives an orthonormal system in $L^2(\mathbb{R})$.

A first natural question is whether or not such functions do exist. Let us observe that taking $\psi = \chi_{(0,1/2)} - \chi_{(1/2,1)}$, the family $\{\psi_{jk}\}$ is the well known Haar system that is in fact a basis for $L^2(\mathbb{R})$. If we want to have a wavelet ψ, smooth and with some decay at infinity, the examples are not that easy. Daubechies and Meyer, among others, constructed wavelets with both properties through a method called a multiresolution analysis.

One of the main advantages in analyzing functions by means of wavelets rather than through the Fourier method, is that it makes possible to obtain characterizations of most of the useful function spaces in terms of wavelet coefficients.

Here we will not give details on what a multiresolution analysis is. For those knowing this method for constructing wavelets, we say that we will be working with a ψ coming from a scale-function ϕ satisfying

(i) $\phi \in C^1(\mathbb{R})$, and

(ii) $|\phi(x)| + |\phi'(x)| \leq C\,(1 + |x|)^{-A}$, with $A > 3$.

Given a function f, its coefficients with respect to the system ψ_{jk} are given by

$$\langle f, \psi_{jk}\rangle = \int f(t)\psi_{jk}(t)\,dt.$$

Although any function in L^2 can be described in terms of coefficients derived from any basis, if we have a system coming from a "good" wavelet, that result can also be extended to L^p, $1 < p < \infty$. In that case we obtain

$$\|f\|_p \simeq \Big\| \Big(\sum |\langle f, \psi_{jk}\rangle|^2\, \chi_{I_{jk}}\, |I_{jk}|^{-1} \Big)^{1/2} \Big\|_p,$$

where I_{jk} denotes the interval $\big[k\,2^{-j}, (k+1)\,2^{-j}\big]$.

Also, the Besov seminorm of a function can be described in terms of wavelet coefficients. For the application we have in mind we shall need the following result (for a proof see [W, p. 228]).

Theorem 9. *Let ψ be a wavelet associated to a multiresolution analysis satisfying (i) and (ii). Assume further that $|\psi(x)| \le c\,(1+|x|)^{-A}$. Then, for $0<\alpha<1$ and $1 \le p, s \le \infty$ there exists a constant C such that*

$$\Bigg(\sum_{j\in\mathbb{Z}} \Big[2^{j\alpha}\,\big(\sum_k 2^{jp\,(1/2-1/p)}\,|\langle f,\psi_{jk}\rangle|^p\big)^{1/p}\Big]^s\Bigg)^{1/s} \le C\,\|f\|_{p,\alpha,s}.$$

We are interested in the case $s = p = \left(\alpha+\frac{1}{2}\right)^{-1}$ with $\alpha \le 1/2$. In that situation the above theorem establishes

$$\sum_{j,k} |\langle f,\psi_{jk}\rangle|^p \le C\,\|f\|_{p,\alpha,p}^p.$$

Assuming this result, it is our intention to investigate the following problem in data compression.

Suppose we have a function $f \in L^2$. We know in this case that f may be approximated in the L^2-norm by a finite sum of its expansion, $\sum_{jk}\langle f,\psi_{jk}\rangle\,\psi_{jk}$. Now, assume that we can keep records of only a fix number of coefficients N, not necessarily the first ones. How good is this approximation measured in the L^2 norm? In other words: can we express the order of the approximation in terms of N for all functions in L^2?

The following example shows that the answer is negative if we deal with a general function in L^2. In fact, suppose we are allowed to use N coefficients to approximate f, i.e., we search for the best approximation of f, in the sense of L^2, by $\sum_{(j,k)\in A}\langle f,\psi_{jk}\rangle\,\psi_{jk}$, where $A \subset \mathbb{Z}\times\mathbb{Z}$, and $\mathrm{card}(A) \le N$.

Let $f_N = \sum_{k=1}^{2N}\frac{1}{\sqrt{2N}}\,\psi_{0,k}$. Then $\|f_N\|_2 = 1$, and for any A with $\mathrm{card}(A) \le N$ we have

$$\Big\|f_N - \sum_{(j,k)\in A}\langle f,\psi_{jk}\rangle\,\psi_{jk}\Big\|_2 \ge \frac{1}{\sqrt{2}}.$$

This is so because the best choice for A is to keep non-vanishing coefficients and having only $2N$ of them and with the same size, we may choose for example $A = \{(0,k) : k = 1,\ldots,N\}$, and the norm of the difference gives in this case $\left(\sum_{N+1}^{2N} 1/(2N)\right)^{1/2} = 1/\sqrt{2}$.

Since the L^2-norm of a function is the ℓ^2-norm of its coefficients taken with respect to an orthonormal basis, we may think the above problem in the following way.

Given a sequence $a = \{a_k\}_{k\in\mathbb{Z}}$, with $\|a\|_{\ell^2} = 1$, and a natural number N: what additional conditions on the sequence would guarantee that choosing the N largest coefficients (in absolute value) we will get a “good” approximation of the original one? (“good” means here that the error goes to zero with N, or better yet, that goes to zero like a negative power of N).

Let us define the set $B \subset \mathbb{Z}$ such that $\operatorname{card}(B) = N$ and $|a_k| \geq |a_\ell|$ whenever $k \in B$ and $\ell \notin B$. Let b the sequence defined by

$$b_k = \begin{cases} a_k & \text{if } k \in B, \\ 0 & \text{if } k \notin B. \end{cases}$$

Assume further that $a \in \ell^p$ for some p, $1 \leq p < 2$ with $\|a\|_{\ell^p} \leq C$. Then we have

$$\sup_{k \notin B} |a_k| \leq \min_{k \in B} |a_k| \leq \Big(\frac{1}{N} \sum_{k \in B} |a_k|^p \Big)^{1/p} \leq C\, N^{-1/p}.$$

Since $p < 2$, it follows that

$$\begin{aligned} \|b - a\|_{\ell^2} = \Big(\sum_{k \notin B} |a_k|^2 \Big)^{1/2} &= \Big(\sum_{k \notin B} |a_k|^{2-p}\, |a_k|^p \Big)^{1/2} \\ &\leq \big(C\, N^{-1/p} \big)^{1-p/2}\, C^{p/2} = C\, N^{1/2 - 1/p}, \end{aligned}$$

and we obtain a "good" approximation since $1/2 - 1/p < 0$.

We may rephrase what we have done in the following way. Assume as above $a \in \ell^2 \cap \ell^p$, with $p < 2$, $\|a\|_{\ell^2} = 1$, and $\|a\|_{\ell^p} \leq C$. Instead of fixing N, we fix a lower threshold for the size of the coefficients, say δ with $\delta > 0$, and let us approximate by the sequence neglecting those coefficients less than δ. Let now $B = \{k : |a_k| \geq \delta\}$ and define the sequence b as above. The previous estimates give that

$$\operatorname{card}(B) \leq \frac{C^p}{\min_{k \in B} |a_k|^p} \leq C^p\, \delta^{-p},$$

and hence

$$\|b - a\|_{\ell^2} \leq C^{p/2}\, \delta^{1-p/2}.$$

In this way the approximation improves as $\delta \to 0$, and the velocity of convergence increases when p gets closer to 1.

Coming back to wavelet expansions, the above discussion shows that although such non linear approximation methods may not be good for all the functions in L^2, they will work for some special subspaces, namely those functions satisfying

$$\Big(\sum_{jk} |\langle f, \psi_{jk} \rangle|^p \Big)^{1/p} \leq C$$

for some $p < 2$.

The description of Besov spaces in terms of wavelet coefficients allows us to conclude that they are the appropriate spaces to make these methods converge. The precise result is the following.

Theorem 10. *Let* $0 < \alpha \le 1/2$, $p = \left(\alpha + \frac{1}{2}\right)^{-1}$, *and* $f \in L^2 \cap \dot{B}^p_{\alpha,p}$. *Then, there exists a constant* K *such that for any* $N \in \mathbb{N}$ *it is possible to find a set* A, $A \subset \mathbb{Z} \times \mathbb{Z}$, *with* $\operatorname{card}(A) = N$ *and*

$$\Big\| f - \sum_{(j,k)\in A} \langle f, \psi_{jk} \rangle \, \psi_{jk} \Big\|_2 \le K \, \|f\|_{p,\alpha,p} \, N^{-\alpha}.$$

Or, alternatively, there exists a constant M *such that for any* $\delta > 0$, *if* $B_\delta = \{(j,k) : |\langle f, \psi_{jk} \rangle| \ge \delta\}$, *we have*

$$\Big\| f - \sum_{(j,k)\in B_\delta} \langle f, \psi_{jk} \rangle \, \psi_{jk} \Big\|_2 \le M \, \|f\|_{p,\alpha,p} \, \delta^{2\alpha}.$$

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