The Banach space L_p

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Our goal is to explore the structure of the "small" subspaces of L_p, mainly for $2 < p < \infty$, discussing older classical results and ultimately presenting some new results of [HOS]. We will review first some Banach space basics. By L_p we shall mean $L_p[0,1]$, under Lebesgue measure m.

Unless we say otherwise $X, Y, \ldots$ shall denote a separable infinite dimensional Banach space. $X \subseteq Y$ means that X is a closed subspace of Y. $X \overset{C}{\sim} Y$ means that X is *C-isomorphic to* Y, i.e., there exits an invertible bounded linear $T : X \to Y$ with $\|T\|\,\|T^{-1}\| \le C$. If$X \overset{1}{\sim} Y$ we shall say X is *isometric* to Y. $X \overset{C}{\hookrightarrow} Y$ means X is C-isomorphic to a subspace of Y.

Definition. A *basis* for X is a sequence $(x_i)_1^\infty \subseteq X$ so that for all $x \in X$ there exists a unique sequence $(a_i) \subseteq \mathbb{R}$ with $x = \sum_1^\infty a_i x_i$, i.e., $\lim_n \sum_{i=1}^n a_i x_i = x$.

Example. The unit vector basis $(e_i)_{i=1}^\infty$ is a basis for ℓ_p $(1 \le p < \infty)$. Of course $e_i = (\delta_{i,j})_{j=1}^\infty$ where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

Definition. $(x_i)_1^\infty \subseteq X$ is *basic* if $(x_i)_1^\infty$ is a basis for $[(x_i)] \equiv$ the closed linear span of $(x_i)_1^\infty$.

Proposition 1. *Let $(x_i)_1^\infty \subseteq X$. Then*
1) *(x_i) is basic iff $x_i \ne 0$ for all i and for some $K < \infty$, all $n < m$ in $\mathbb{N}$ and all $(a_i)_1^m \subseteq \mathbb{R}$,*

$$\Big\|\sum_1^n a_i x_i\Big\| \le K \Big\|\sum_1^m a_i x_i\Big\| \ .$$

(In this case (x_i) is called K-basic *and the smallest K satisfying* 1) *is called the* basis constant *of (x_i).)*
2) *(x_i) is a basis for X iff* 1) *holds and $[(x_i)] = X$.*

(x_i) is called *monotone* if its basis constant is 1.

The proof of this and other background facts we present can be found in any of the standard texts such as [LT1], [AK], [D], [FHHSPZ]. The paper [AO] contains further background on L_p spaces.

Definition. A bounded linear operator $P : X \to X$ is a *projection* if $P^2 = P$.

In this case if $Y = P(X)$ then $X = Y \oplus \operatorname{Ker} P$. Writing $X = Y \oplus Z$ means that Y and Z are closed subspaces of X and every $x \in X$ can be uniquely written $x = y + z$ for some $y \in Y$, $z \in Z$. In this case $Px = y$ defines a projection of X onto Y. Y is said to be *complemented* in X if it is the range of a projection on X. Y is *C-complemented* in X if $\|P\| \le C$.

If $F \subseteq X$ is a finite dimensional subspace then F is complemented in X. If X is isomorphic to ℓ_2 then all $Y \subseteq X$ are complemented but this fails to be the case if $X \not\sim \ell_2$ by a result of Lindenstrauss and Tzafriri [LT2].

Now from Propostion 1 if (x_i) is a basis for X then setting $P_n(\sum a_i x_i) = \sum_{i=1}^n a_i x_i$ yields a projection of X onto $\langle (x_i)_1^n \rangle \equiv$ linear span of $(x_i)_{i=1}^n$. Moreover the P_n's are uniformly bounded and $\sup_n \|P_n\|$ is the basis constant of (x_i).

Not every Banach space X has a basis but the standard ones do.

The Haar basis for L_p $(1 \le p < \infty)$: The Haar basis $(h_i)_1^\infty$ is a monotone basis for L_p.

$$\begin{aligned}
&h_1 \equiv 1 \\
&h_2 = \mathbf{1}_{[0,\,1/2]} - \mathbf{1}_{[1/2,\,1]} \\
&h_3 = \mathbf{1}_{[0,\,1/4]} - \mathbf{1}_{[1/4,\,1/2]}\ , \qquad h_4 = \mathbf{1}_{[1/2,\,3/4]} - \mathbf{1}_{[3/4,\,1]} \\
&\quad \dots
\end{aligned}$$

To see this is a monotone basis for L_p is not hard via Proposition 1. We need only check a couple of things. First

$$\langle (h_i)_1^{2^n} \rangle = \Big\{ f = \sum_1^{2^n} a_i \mathbf{1}_{D_i^n} : (a_i)_1^{2^n} \subseteq \mathbb{R} \Big\} \ \text{ where } \ D_i^n = \Big[\frac{i-1}{2^n}, \frac{i}{2^n} \Big]\ .$$

From real analysis these functions (over all n) are dense in L_p $(1 \le p < \infty)$.

Secondly to see 1) holds with $K = 1$ it suffices to show for all n, $(a_i)_1^{n+1} \subseteq \mathbb{R}$,

$$\Big\| \sum_1^n a_i h_i \Big\|_p \le \Big\| \sum_1^{n+1} a_i h_i \Big\|_p\ .$$

This reduces to proving if $D = [\frac{i-1}{2^j}, \frac{i}{2^j}]$ is a dyadic interval with left half D_+ and right half D_- supporting the Haar function $h = \mathbf{1}_{D_+} - \mathbf{1}_{D_-}$ then for all $a, b \in \mathbb{R}$, $\|a\mathbf{1}_D\|_p = \|a\mathbf{1}_D + h\|_p$ or

$$\|a\mathbf{1}_D\|_p \le \|(a+b)\mathbf{1}_{D_+} + (a-b)\mathbf{1}_{D_-}\|_p\ .$$

This is an easy exercise.

Definition. Basic sequences (x_i) and (y_i) are *C-equivalent* if there exist A, B with $A^{-1}B \le C$ and for all $(a_i) \subseteq \mathbb{R}$

$$\frac{1}{A} \Big\| \sum a_i y_i \Big\| \le \Big\| \sum a_i x_i \Big\| \le B \Big\| \sum a_i y_i \Big\|\ .$$

This just says that the linear map $T : [(x_i)] \to [(y_i)]$ with $Tx_i = y_i$ for all i is an onto isomorphism with $\|T\|\,\|T^{-1}\| \le C$.

Proposition 2 (Perturbations). *Let (x_i) be a normalized K-basic sequence in X and let $(y_i) \subseteq X$ satisfy*

$$\sum_{i=1}^\infty \|x_i - y_i\| \equiv \lambda < \frac{1}{2K}\ .$$

Then (x_i) is $C(\lambda)$-equivalent to (y_i) where $C(\lambda) \downarrow 1$ as $\lambda \downarrow 0$. If in addition $[(y_i)]$ is complemented in X by a projection P and $\lambda < \frac{1}{8K\|P\|}$ then $[(x_i)]$ is complemented in X by a projection Q where $\|Q\| \to \|P\|$ as $\lambda \downarrow 0$.

Notation. If (x_i) and (y_i) are C-equivalent basic sequences we write $(x_i) \overset{C}{\sim} (y_i)$.

Definition. Let (x_i) be basic. (y_i) is a *block basis* of (x_i) if $y_i \neq 0$ for all i and for some $0 = n_0 < n_1 < n_2 < \cdots$ and $(a_i) \subseteq \mathbb{R}$, $y_i = \sum_{j=n_{i-1}+1}^{n_i} a_j x_j$.

Note. (y_i) is then automatically basic with basis constant not exceeding that of (x_i).

If (x_i) is a normalized K-basis for X we define the *coordinate* or *biorthogonal* functionals (x_i^*) via $x_i^*(\sum a_j x_j) = a_i$. From Proposition 1 we obtain $\|x_i^*\| \leq 2K$ and so for all (a_i)

$$\frac{1}{2K}\|(a_i)\|_\infty \leq \Big\| \sum a_i x_i \Big\| \leq \sum |a_i| = \|(a_i)\|_{\ell_1} \ .$$

In other words $\|\sum a_i x_i\|$ is trapped between the c_0 and ℓ_1 norms of (a_i).

From Proposition 2 we obtain

Proposition 3. *Let X have a basis (x_i) and let $(y_i) \subset S_X \equiv \{x \in X : \|x\| = 1\}$ be weakly null (i.e., $x^*(y_i) \to 0$ for all $x^* \in X^*$). Then given $\varepsilon_i \downarrow 0$ there exists a subsequence (z_i) of (y_i) and a block basis $(b_i) \subseteq S_X$ of (x_i) with $\|z_i - b_i\| < \varepsilon_i$ for all i. In particular given $\varepsilon > 0$ we can choose (z_i) to be $(1+\varepsilon)$-equivalent to a normalized block basis of (x_i).*

Definition. A basis (x_i) for X is K-*unconditional* if for all $\sum a_i x_i \in X$ and all $\varepsilon_i = \pm 1$,

$$\Big\| \sum a_i x_i \Big\| \leq K \Big\| \sum \varepsilon_i a_i x_i \Big\| \ .$$

It is not hard to show (x_i) is unconditional iff for all $x = \sum a_i x_i \in X$ and all permutations π of $\mathbb{N}$,

$$x = \sum a_{\pi(i)} x_{\pi(i)} \ .$$

iff for some $C < \infty$, all $\sum a_i x_i \in X$ and all $M \subseteq \mathbb{N}$, $\|\sum_{i \in M} a_i x_i\| \leq C \|\sum a_i x_i\|$. (This just says that the projections $(P_M : M \subseteq \mathbb{N})$ given as above are well defined and uniformly bounded.)

Easily, the unit vector basis (e_i) is a 1-unconditional basis for ℓ_p $(1 \leq p < \infty)$ or c_0.

Fact. The Haar basis is an unconditional basis for L_p if $1 < p < \infty$.

This is a more difficult result (see [Bu1]), if $p \neq 2$. For $p = 2$, (h_i) is an orthogonal basis

$$\Big\| \sum a_i \frac{h_i}{\|h_i\|_2} \Big\|_2 = \Big(\sum |a_i|^2 \Big)^{1/2} \ .$$

More generally if (x_i) is a normalized block basis of (h_i) then $\|\sum a_i x_i\|_2 = (\sum |a_i|^2)^{1/2}$.

It is easy to check that (h_i) is not unconditional in L_1. For example if

$$(y_i) = (h_1, h_2, h_3, h_5, h_9, h_{17}, \ldots)$$

is the sequence of "left most" h_i's then

$$\Big\|\sum_1^n \frac{y_i}{\|y_i\|_1}\Big\|_1 = 1 \ \text{ while for some } \ c > 0\ , \quad \Big\|\sum_1^n (-1)^i \frac{y_i}{\|y_i\|_1}\Big\|_1 \geq cn\ .$$

Definition. A *finite dimensional decomposition* (FDD) for X is a sequence of non-zero finite dimensional subspaces (F_i) of X so that for all $x \in X$ there exists a unique sequence (x_i) with $x_i \in F_i$ for all i and $x = \sum x_i$.

As with bases the projections $P_n x = P_n(\sum x_i) = \sum_1^n x_i$ are uniformly bounded and $\sup_n \|P_n\|$ is the basis constant of the FDD. Also for $n \leq m$ if $P_{[n,m]}x = \sum_n^m x_i$, then the $P_{[n,m]}$'s are uniformly bounded and $\sup_{n\leq m} \|P_{[n,m]}\|$ is the *projection constant* of the FDD. (E_i) is monotone if its basis constant is 1 and *bimonotone* if its projection constant is 1.

A *blocking* (G_i) of an FDD (F_i) for X is given by $G_i = \langle (F_j)_{j=n_{i-1}+1}^{n_i}\rangle$ for some $0 = n_0 < n_1 < \cdots$. (G_i) is then also an FDD for X.

A basis (x_i) also may be regarded as an FDD with $F_i = \langle x_i\rangle$.

From Proposition 3 we see that if $1 < p < \infty$ and $(y_i) \subseteq S_{L_p}$ is weakly null (equivalently, $\int_E y_i \to 0$ for all measurable $E \subseteq [0,1]$) then some subsequence is a perturbation of a block basis of (h_i) and hence is unconditional (just like for bases, block bases of unconditional bases are unconditional). This fails in L_1 by a deep new result of Johnson, Maurey and Schechtman.

Theorem 4. [JMS] *There exists a weakly null sequence $(x_i) \subseteq S_{L_1}$ satisfying: for all $\varepsilon > 0$ and all subsequences $(y_i) \subseteq (x_i)$, (h_i) is $(1+\varepsilon)$-equivalent to a block basis of (y_i).*

Now lets fix $2 < p < \infty$ and let K_p be the unconditional constant of (h_i) in L_p. We shall list what we consider to be the small subspaces of L_p. These are also subspaces of L_p for $1 < p < 2$ but as we shall note shortly the situation there as to what constitutes "small" is more complicated.

L_p contains the following "small" subspaces

- ℓ_p (isometrically): If $(x_i) \subseteq S_{L_p}$ are disjointly supported then

$$\begin{aligned}\|\sum a_i x_i\| &= \Big(\int |\sum a_i x_i(t)|^p\, dt\Big)^{1/p} \\ &= \Big(\sum \int |a_i|^p |x_i(t)|^p\, dt\Big)^{1/p} \\ &= \Big(\sum |a_i|^p\Big)^{1/p}.\end{aligned}$$

Also $[(x_i)]$ is 1-complemented in X via $Px = \sum_{i=1}^\infty x_i^*(x) x_i$ where (x_i^*) are the functions naturally biorthogonal to (x_i), $x_i^* = \operatorname{sign}(x_i)|x_i|^{p-1}$.

- ℓ_2 (isomorphically) via the Rademacher functions (r_n). (r_n) are ± 1 valued independent random variables of mean 0.

Khintchin's inequality: For $2 < p < \infty$,

$$\begin{aligned}\Big(\sum |a_n|^2\Big)^{1/2} &= \big\|\sum a_n r_n\big\|_2 \le \big\|\sum a_n r_n\big\|_p \\ &\le B_p\Big(\sum |a_n|^2\Big)^{1/2}.\end{aligned}$$

For $1 < p < 2$

$$A_p\Big(\sum |a_n|^2\Big)^{1/2} \le \|\sum a_n r_n\|_p \le \|\sum a_n r_n\|_2 = \Big(\sum |a_n|^2\Big)^{1/2}.$$

The constants A_p, B_p depend solely on p.

- ℓ_2 (isometrically) via a sequence of symmetric Gaussian independent random variables in S_{L_p}
- $(\ell_2 \oplus \ell_p)_p$ (isometrically)

For this we use that $L_p \overset{1}{\sim} (L_p[0,\frac12] \oplus L_p[\frac12,1])_p$ and $L_p[0,\frac12] \overset{1}{\sim} L_p[\frac12,1] \overset{1}{\sim} L_p[0,1]$. More generally if we partition $[0,1]$ into disjoint intervals of positive measure $(I_n)_{n=1}^\infty$ then $L_p(I_n) \overset{1}{\sim} L_p$ and $L_p \overset{1}{\sim} (\sum L_p(I_n))_p$. Hence L_p contains also

- $(\sum \ell_2)_p = (\ell_2 \oplus \ell_2 \oplus \cdots)_p \equiv \{(x_i) : x_i \in \ell_2$ for all i and $\|(x_i)\| = (\sum \|x_i\|_2^p)^{1/p} < \infty\}$ (isometrically)

Our topic will be to characterize when $X \subseteq L_p$, $2 < p < \infty$, embeds isomorphically into or contains isomorphically one of the four spaces ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$ or $(\sum \ell_2)_p$.

Now some remarks are in order here. First it is known that $L_q \overset{1}{\hookrightarrow} L_p$ if $p < q \le 2$ ($X \overset{C}{\hookrightarrow} Y$ means X is C-isomorphic to a subspace of Y). Thus L_p contain ℓ_q if $p < q < 2$ so is this "small"? Secondly we have

Proposition 5. *Let $X \subseteq \ell_p$ $(1 \le p < \infty)$. Then for all $\varepsilon > 0$ there exists $Y \subseteq X$ with $Y \overset{1+\varepsilon}{\sim} \ell_p$ and Y is $1+\varepsilon$-complemented in ℓ_p.*

This is due to Pełczyński [P]. Every normalized block basis of (e_i) in ℓ_p is 1-equivalent to (e_i) and 1-complemented in ℓ_p as is easily checked. Then one uses perturbation as in Proposition 2.

Some other classical facts are

i) The ℓ_p spaces are totally incomparable, i.e., for all $X \subseteq \ell_p$, $Y \subseteq \ell_q$, $p \ne q$, $X \not\sim Y$.

ii) For $1 \le p, q < \infty$, $L_q \hookrightarrow L_p$ iff $q = 2$ or $1 \le p \le q < 2$. Also $\ell_q \hookrightarrow L_p$ iff $1 \le p \le q < 2$ or $q = 2$.

Our next result shows that normalized unconditional basic sequences in L_p, $1 < p < \infty$, are trapped between the ℓ_p and ℓ_2 norms.

Proposition 6. a) *Let $2 < p < \infty$ and let $(x_i) \subseteq S_{L_p}$ be λ-unconditional. Then for all $(a_n) \subseteq \mathbb{R}$,*

$$\lambda^{-1}\Big(\sum |a_n|^p\Big)^{1/p} \le \big\|\sum a_n x_n\big\|_p \le \lambda B_p\Big(\sum |a_n|^2\Big)^{1/2} .$$

b) *Let $1 < p < 2$ and let $(x_i) \subseteq S_{L_p}$ be λ-unconditional. Then for all $(a_i) \subseteq \mathbb{R}$,*

$$(\lambda A_p)^{-1}\Big(\sum |a_n|^2\Big)^{1/2} \le \big\|\sum a_n x_n\big\|_p \le \lambda\Big(\sum |a_n|^p\Big)^{1/p} .$$

Proof. For $t \in [0,1]$, $2 < p < \infty$,

$$\big\|\sum a_n x_n\big\|_p \le \lambda\big\|\sum a_n x_n r_n(t)\big\|_p$$

and so

$$\begin{aligned}
\big\|\sum a_n x_n\big\|_p^p &\le \lambda^p \int_0^1 \big\|\sum a_n x_n r_n(t)\big\|_p^p \, dt \\
&\overset{\text{(Fubini)}}{=} \lambda^p \int_0^1 \int_0^1 \big|\sum a_n x_n(s) r_n(t)\big|^p \, dt\, ds \\
&\le (\lambda B_p)^p \int_0^1 \Big(\sum a_n^2 x_n(s)^2\Big)^{p/2} ds \\
&\le (\lambda B_p)^p \Big(\sum \|a_n^2 x_n^2\|_{p/2}\Big)^{p/2}
\end{aligned}$$

(by the triangle inequality in $L_{p/2}$)

$$= (\lambda B_p)^p \Big(\sum |a_n|^2\Big)^{p/2} .$$

This gives the upper ℓ_2-estimate.

Similarly,

$$\begin{aligned}\lambda^p \Big\| \sum a_n x_n \Big\|^p &\geq \int_0^1 \Big(\sum a_n^2 x_n^2(s) \Big)^{p/2} ds \\ &\geq \int_0^1 \sum |a_n|^p |x_n(s)|^p \, ds = \sum |a_n|^p\end{aligned}$$

(using $\|\cdot\|_{\ell_p} \leq \|\cdot\|_{\ell_2}$). The argument is similar for $1 < p < 2$. □

The technique of proof, integrating against the Rademacher functions, yields

Proposition 7. *For $1 < p < \infty$ there exists $C(p)$ so that if $(x_i) \subseteq S_{L_p}$ is λ-unconditional then for all (a_i)*

$$\Big\| \sum a_n x_n \Big\|_p \overset{\lambda C(p)}{\sim} \left(\int_0^1 \Big(\sum |a_n|^2 |x_n(s)|^2 \Big)^{p/2} ds \right)^{1/p} . \tag{1}$$

The expression on the right is the so called "square function." By $A \overset{C}{\sim} B$ we mean $A \leq CB$ and $B \leq CA$.

Corollary 8. [S2] *Let $(x_n) \subseteq S_{L_p}$, $1 < p < \infty$, be unconditional basic. Then (x_n) is equivalent to a block basis (y_n) of (h_n).*

Sketch. By (1) it follows that if (y_i) is a block basis of (h_i) with $|y_i| = |x_i|$ on $[0,1]$ then $(y_i) \sim (x_i)$. By a perturbation argument we may assume each $x_i \in \langle h_j \rangle$. Then it is easy to construct the y_i's. Indeed given a simple dyadic function x and any n one can find $y \in \langle h_i \rangle_n^\infty$ so that $|y| = |x|$. □

We are now ready to begin our investigation announced previously: if $X \subseteq L_p$ $(2 < p < \infty)$ when does X contain or embed into one of the 4 small subspaces of L_p, namely ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$ or $(\sum \ell_2)_p$? We begin with a result from 1960.

Theorem 9 (Kadets and Pełczyński [KP])**.** *Let $X \subseteq L_p$, $2 < p < \infty$. Then $X \sim \ell_2$ iff $\|\cdot\|_2 \sim \|\cdot\|_p$ on X; i.e., for some C, $\|x\|_2 \leq \|x\|_p \leq C\|x\|_2$ for all $x \in X$. Moreover there is a projection $P : L_p \to X$.*

Sketch. First note that if $x \in S_{L_p}$ and $m\{t : |x(t)| \geq \varepsilon\} \geq \varepsilon$ then $\|x\|_2 \leq \|x\|_p = 1 \leq \varepsilon^{-3/2}\|x\|_2$. Indeed

$$\|x\|_2 = \left(\int |x(t)|^2 \, dt \right)^{1/2} \geq \left(\int_{[|x| \geq \varepsilon]} |x(t)|^2 \, dt \right)^{1/2} \geq \varepsilon \cdot \varepsilon^{1/2} .$$

The direction requiring proof is if $X \sim \ell_2$ then $\|\cdot\|_2 \sim \|\cdot\|_p$ on X. If not we can find $(x_i) \subseteq S_X$, $x_i \xrightarrow{\omega} 0$, so that for all $\varepsilon > 0$, $\lim_n m[|x_n| \geq \varepsilon] = 0$. From this we can construct a subsequence (x_{n_i}) and disjointly supported $(f_i) \subseteq S_{L_p}$ with $\lim_i \|x_{n_i} - f_i\| = 0$. Hence by a perturbation argument a subsequence of (x_i) is equivalent to the unit vector basis of ℓ_p which contradicts $X \sim \ell_2$.

The projection onto X with $\|x\|_p \le C\|x\|_2$ for $x \in X$ is given by the orthogonal projection $P : L_2 \to X$ acting on L_p. For $y \in L_p$,

$$\|Py\|_p \le C\|Py\|_2 \le C\|y\|_2 \le C\|y\|_p \ .$$

□

Remarks. The proof yields that if $X \subseteq L_p$, $2 < p < \infty$, and $X \not\sim \ell_2$ then for all $\varepsilon > 0$, $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$. Moreover if $(x_i) \subseteq S_{L_p}$ is weakly null and $\varepsilon = \lim_i \|x_i\|_2$ then a subsequence is equivalent to the ℓ_p basis if $\varepsilon = 0$ and the ℓ_2 basis if $\varepsilon > 0$.

In the latter case we have essentially (assuming say (x_i) is a normalized block basis of (h_i) with $\|x_i\|_2 = \varepsilon$ for all i)

$$\varepsilon\Big(\sum a_i^2\Big)^{1/2} = \Big\|\sum a_i x_i\Big\|_2 \le \Big\|\sum a_i x_i\Big\|_p \le K_p B_p\Big(\sum a_i^2\Big)^{1/2} \ .$$

Pełczyński and Rosenthal [PR] proved that if $X \overset{K}{\sim} \ell_2$ then X is $C(K)$-complemented in L_p via a change of density argument.

Our next result shows that if X does not contain an isomorph of ℓ_2 then it embeds into ℓ_p. The argument uses "Pełczyński's decomposition method."

Proposition 10. [P] *Let X be a complemented subspace of ℓ_p, $1 \le p < \infty$. Then $X \sim \ell_p$.*

Proof. $\ell_p \sim X \oplus V$ for some $V \subseteq \ell_p$. Also $X \sim \ell_p \oplus W$ for some $W \subseteq X$ by Proposition 5. Finally $\ell_p \sim \ell_p \oplus \ell_p$ and moreover $\ell_p \sim (\ell_p \oplus \ell_p \oplus \cdots)_p$. The latter is proved by splitting (e_i) into infinitely many infinite subsets. Thus

$$\begin{aligned} \ell_p &\sim (\ell_p \oplus \ell_p \oplus \cdots)_p \sim ((X \oplus V) \oplus (X \oplus V) \oplus \cdots)_p \\ &\sim (X \oplus X \oplus \cdots)_p \oplus (V \oplus V \oplus \cdots)_p \\ &\sim X \oplus (X \oplus X \oplus \cdots)_p \oplus (V \oplus V \oplus \cdots)_p \\ &\sim X \oplus \ell_p \sim W \oplus \ell_p \oplus \ell_p \sim W \oplus \ell_p \sim X \ . \end{aligned}$$

□

A consequence of this is that if (H_n) is any blocking of (h_i) into an FDD then $(\sum H_n)_p \sim \ell_p$. Indeed each H_n is uniformly complemented in $\ell_p^{m_n}$ for some m_n, hence $(\sum H_n)_p$ is complemented in $(\sum \ell_p^{m_n})_p = \ell_p$.

Theorem 11. [JO1] *Let $2 < p < \infty$, $X \subseteq L_p$. Then $X \hookrightarrow \ell_p \Leftrightarrow \ell_2 \not\hookrightarrow X$. (*[KW] *If $\ell_2 \not\hookrightarrow X$ then for all $\varepsilon > 0$, $X \overset{1+\varepsilon}{\hookrightarrow} \ell_p$.)*

The scheme of the argument is to show if $\ell_2 \not\hookrightarrow X$ then there is a blocking (H_n) of the Haar basis into an FDD so that $X \hookrightarrow (\sum H_n)_p$ in a natural way; $x = \sum x_n$, $x_n \in H_n \to (x_n) \in (\sum H_n)_p$. Since $(\sum H_n)_p \sim \ell_p$ we are done.

We won't discuss the specifics here of this argument but rather will sketch shortly the proof of a stronger result. First we note the analogous theorem for $1 < p < 2$, which has a different form. Note the Theorem would also hold for $2 < p < \infty$ and, unlike $1 < p < 2$, the constant K need not be specified.

Theorem 12. [Jo] *Let $X \subseteq L_p$, $1 < p < 2$. Then $X \hookrightarrow \ell_p$ if (and only if) there exists $K < \infty$ so that for all weakly null $(x_i) \subseteq S_X$ some subsequence is K-equivalent to the unit vector basis of ℓ_p.*

These results were unified using the infinite asymptotic game/weakly null trees machinery which we will discuss after stating

Theorem 13. *Let $X \subseteq L_p$, $1 < p < \infty$. Then $X \hookrightarrow \ell_p$ iff every weakly null tree in S_X admits a branch equivalent to the unit vector basis of ℓ_p.*

A tree in S_X is $(x_\alpha)_{\alpha \in T_\infty} \subseteq S_X$ where

$$T_\infty = \{(n_1, \ldots, n_k) : k \in \mathbb{N}\ ,\ n_1 < \cdots < n_k \text{ are in } \mathbb{N}\}\ .$$

A *node* in T_∞ is all $(x_{(\alpha,n)})_{n > n_k}$ where $\alpha = (n_1, \ldots, n_k)$ or $\alpha = \emptyset$. The tree is *weakly null* means each node is a weakly null sequence. A *branch* is $(x_i)_{i=1}^\infty$ given by $x_i = x_{(n_1,\ldots,n_i)}$ for some subsequence (n_i) of $\mathbb{N}$.

It is worth noting that, just as in Proposition 3, if $X \subseteq Z$, a space with a basis (z_i) and $(x_\alpha)_{\alpha \in T_\infty} \subseteq S_X$ is a weakly null tree then the tree admits a full subtree $(y_\alpha)_{\alpha \in T_\infty}$ so that each branch is a perturbation of a block basis of (z_i). By *full subtree* we mean that $(y_\alpha)_{\alpha \in T_\infty} = (x_\alpha)_{\alpha \in T'}$ where $T' \subseteq T_\infty$ is order isomorphic to T and if $y_\alpha = x_{\gamma(\alpha)}$ then $|\gamma(\alpha)| = |\alpha| =$ length of α. $|(n_1, \ldots, n_k)| = k$.

Remarks. The conditions for a general reflexive X,

A) Every weakly null sequence $(x_i) \subseteq X$ has a subsequence K-equivalent to the unit vector basis of ℓ_p and

B) Every weakly null tree in S_X admits a branch equivalent to the unit vector basis of ℓ_p are generally different. It is not hard to show that B) actually implies

B)$'$ For some C every weakly null tree in S_X admits a branch C-equivalent to the unit vector basis of ℓ_p.

Also B)$'$ $\Rightarrow$ A) by considering the tree $(x_\alpha)_{\alpha\in T_\alpha}$ where $x_{(n_1,\dots,n_k)} = x_{n_k}$. Indeed the branches of (x_α) coincide with the subsequences of (x_i). But in L_p one can show that A) and B) are in fact equivalent. Thus Theorem 13 encompasses both Theorems 11 and 12.

Theorem 13 follows from

Theorem 14. [OS] *Let $1 < p < \infty$, let X be reflexive and assume that every weakly null tree in S_X admits a branch C-equivalent to the unit vector basis of ℓ_p. Assume $X \subseteq Z$, a reflexive space with an* FDD(E_i)*. Then there exists a blocking (F_i) of (E_i) so that X naturally embeds into $(\sum F_i)_p$.*

The conclusion means that for some K and all $x \in X$, $x = \sum x_n$, $x_n \in F_n$, we have $\|x\| \overset{K}{\sim} (\sum \|x_n\|^p)^{1/p}$.

We shall outline the steps involved in the proof. First we give a definition.

Definition. Let (E_i) be an FDD for Z. Let $\bar\delta = (\delta_i)$, $\delta_i \downarrow 0$. A sequence $(z_i) \subseteq S_Z$ is a *$\bar\delta$-skipped block sequence* w.r.t. (E_i) if there exist integers $1 \le k_1 < \ell_1 < k_2 < \ell_2 < \cdots$ so that

$$\|z_n - P^E_{(k_n,\ell_n]} z_n\| < \delta_n \text{ for all } n .$$

Here for $x = \sum x_i$, $x_i \in E_i$, $P^E_{(k,\ell]}x = \sum_{i\in(k,\ell]} x_i$. Thus above the "skipping" is the $P^E_{k_n}$ terms. (z_n) is almost a block basis of (E_n) with the E_{k_n} almost skipped.

Now let $X \subseteq Z = [(E_i)]$ be as in the statement of Theorem 14.

Step 1. There exists a blocking (G_i) of (E_i) and $\bar\delta$ so that every $\bar\delta$-skipped block sequence w.r.t. (G_i) in S_X is $2C$-equivalent to the unit vector basis of ℓ_p.

To obtain this one first shows that the weakly null tree hypothesis on X is equivalent to (S) having a winning strategy in the following game (for all $\varepsilon > 0$).

> *The infinite asymptotic game*: Two players (S) for subspace and (V) for vector alternate plays forever as follows. (S) chooses $n_1 \in \mathbb{N}$. (V) chooses $x_1 \in S_X \cap [(E_i)_{i\ge n_1}], \dots$. Thus the plays are $(n_1, x_1, n_2, x_2, \dots)$.

(S) wins if $(x_i) \in \mathcal{A}(C+\varepsilon) \equiv \{$all normalized bases $(C+\varepsilon)$-equivalent to the unit vector basis of $\ell_p\}$.

(S) *has a winning strategy* means that

$$\exists\, n_1\ \forall\, x_1 \in S_X \cap [(E_i)_{i\ge n_1}]$$
$$\exists\, n_2\ \forall\, x_2 \in S_X \cap [(E_i)_{i\ge n_2}]$$
$$\dots$$
$$(x_i) \in \mathcal{A}(C+\varepsilon)$$

(V) *wins* if $(x_i) \notin \mathcal{A}(C+\varepsilon)$.

(V) *has a winning strategy* means that

$$\forall\, n_1\ \exists\, x_1 \in S_X \cap [(E_i)_{i\ge n_1}]$$
$$\forall\, n_2\ \exists\, x_2 \in S_X \cap [(E_i)_{i\ge n_2}]$$
$$\dots$$
$$(x_i) \notin \mathcal{A}(C+\varepsilon)$$

Now these two winning strategies are the formal negations of each other, but they are infinite sentences so must one be true? Yes, if the game is determined which it is in this case since Borel games are determined [Ma]. Now if (V) had a winning strategy one could easily produce a weakly null tree in S_X all of whose branches did not lie in $\mathcal{A}(C+\varepsilon)$. So (S) has a winning strategy. Then by a compactness argument one can deduce Step 1 ($2C$ could be any $C+\varepsilon$ here).

The next step is a lemma of W.B. Johnson [Jo] which allows us to decompose any $x \in S_X$ into (almost) a linear combination of $\bar{\delta}$-skipped blocks, in X.

Step 2. Let K be the projection constant of (G_i). There exists a blocking (F_i) of (G_i), $F_i = \langle G_i\rangle_{j\in(N_{i-1},N_i]}$, $N_0 = 0 < N_1 < \cdots$, satisfying the following.

For all $x \in S_X$ there exists $(x_i) \subseteq X$ and for all i there exists $t_i \in (N_{i-1}, N_i)$ ($t_0 = 0$, $t_1 > 1$) satisfying

a) $x = \sum x_j$
b) $\|x_i\| < \delta_i$ or $\|P^G_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i \|x_i\|$
c) $\|P^G_{(t_{i-1},t_i)} x - x_i\| < \delta_i$
d) $\|x_i\| < K+1$
e) $\|P^G_{t_i} x\| < \delta_i$

Moreover the above holds for any further blocking of (G_i) (which redefines the N_i's).

Remark. Thus if $x \in S_X$ we can write $x = \sum x_i$, $(x_i) \subseteq X$ where if $B = \{i : \|x_i\| \geq \delta_i\}$ then $(\frac{x_i}{\|x_i\|})_{i \in B}$ is a $\bar{\delta}$-skipped block sequence w.r.t. (G_i). Also the skipped blocks (G_{t_i}) are in predictable intervals, $t_i \in (N_{i-1}, N_i)$. And $\sum_{i \notin B} \|x_i\| < \sum \delta_i$.

To prove Step 2 we have a

Lemma. *$\forall\, \varepsilon > 0\ \forall\, N \in \mathbb{N}\ \exists\, n > N$ so that if $x \in B_X$, $x = \sum y_i$, $y_i \in G_i$, then there exists $t \in (N, n)$ with*

$$\|y_t\| < \varepsilon \quad \textit{and} \quad \operatorname{dist}\left(\sum_1^{t-1} y_i, X\right) < \varepsilon\ .$$

Proof. If not we obtain $y^{(n)} \in B_X$ for $n > N$ failing the conclusion for $t \in (N, n)$. Choose $y^{(n_i)} \xrightarrow{\omega} y \in B_X$ and let $t > N$ satisfy $\|P^G_{[t,\infty)} y\| < \varepsilon/2K$. Choose $y^{(n)}$ from $(y^{(n_i)})$ so that $n > t$ and $\|P^G_{[1,t)}(y^{(n)} - y)\| < \varepsilon/2K$. Then

$$\|P^G_{[1,t)} y^{(n)} - y\| \leq \|P^G_{[1,t)}(y^{(n)} - y)\| + \|P^G_{[t,\infty)} y\| < \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} \leq \varepsilon\ .$$

Also

$$\|P^G_t y^{(n)}\| \leq \|P^G_t (y^{(n)} - y)\| + \|P^G_t y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

This contradicts our choice of $y^{(n)}$. □

To use the lemma we select $N_0 = 0 < N_1 < N_2 < \cdots$ so that for all $x \in B_X$ there exists $t_i \in (N_{i-1}, N_i)$ and $z_i \in X$ with $\|P^G_{t_i} x\| < \varepsilon_i$ and $\|P^G_{[1,t_i)} x - z_i\| < \varepsilon_i$. Set $x_i = z_1$, $x_i = z_i - z_{i-1}$ for $i > 1$. Then $\sum_1^n x_i = z_n \to x$ and the other properties b)–d) hold, as is easily checked, if $(K+1)(\varepsilon_i + 2\varepsilon_{i-1}) < \delta_i^2$. □

Now let (F_i) be the blocking obtained in Step 2. It is not hard to show that if $x = \sum x_i$ is as in Step 2 then $\|x\| \overset{3C}{\sim} (\sum \|x_i\|^p)^{1/p}$, provided $\bar{\delta} = (\delta_i)$ is small enough. But this is not the decomposition given by $x = \sum y_i$, $y_i \in F_i$. However we do have

$$\begin{aligned} x_i &\approx P^G_{(t_{i-1}, t_i)}(y_{i-1} + y_i) \quad \text{and} \\ y_i &\approx P^G_{(N_{i-1}, N_i)}(x_i + x_{i+1}) \end{aligned}$$

which yields $\|x\| \overset{K(C)}{\sim} (\sum \|y_i\|^p)^{1/p}$ by making the appropriate estimates. □

Returning to $X \subseteq L_p$ $(2 < p < \infty)$ we have seen that one of these holds:

- $X \sim \ell_2$
- $X \hookrightarrow \ell_p$
- $\ell_p \oplus \ell_2 \hookrightarrow X$

The latter comes from Theorems 9 and 11. If $X \not\approx \ell_2$ and $X \not\hookrightarrow \ell_p$ then X contains a subspace isomorphic to ℓ_2 so $X \sim \ell_2 \oplus Y$. Now Y also contains ℓ_p (or else $X \sim \ell_2$) and in fact complementably (as a perturbation of a disjointly supported $(f_i) \subseteq S_{L_p}$) so $\ell_p \oplus \ell_2 \hookrightarrow X$.

Our next goal will be to characterize when $X \hookrightarrow \ell_p \oplus \ell_2$ and if not to then show that $(\sum \ell_2)_p \hookrightarrow X$.

First we recall one more old result.

Theorem 15. [JO2] *Let $X \subseteq L_p$, $2 < p < \infty$. Assume there exists $Y \subseteq \ell_p \oplus \ell_2$ and a quotient (onto) map $Q : Y \to X$. Then $X \hookrightarrow \ell_p \oplus \ell_2$.*

This is an answer, of a sort, to when $X \hookrightarrow \ell_p \oplus \ell_2$ but it is not an intrinsic characterization. The proof however provides a clue as to how to find one. The isomorphism $X \hookrightarrow \ell_p \oplus \ell_2$ is given by a blocking (H_n) of (h_i) so that X naturally embeds into

$$\Big(\sum H_n\Big)_p \oplus \Big(\sum (H_n, \|\cdot\|_2)\Big)_2 \sim \ell_p \oplus \ell_2 \ .$$

Before proceeding we recall some more inequalities.

Theorem 16. [R] *Let $2 < p < \infty$. There exists $K_p < \infty$ so that if (x_i) is a normalized mean zero sequence of independent random variables in L_p then for all $(a_i) \subseteq \mathbb{R}$,*

$$\Big\| \sum a_i x_i \Big\|_p \overset{K_p}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big(\sum |a_i|^2 \|x_i\|_2^2\Big)^{1/2} \ .$$

Note that in this case $[(x_i)] \hookrightarrow \ell_p \oplus \ell_2$ via the embedding

$$\sum a_i x_i \longmapsto \Big((a_i)_i, (a_i \|x_i\|_2)_i\Big) \in \ell_p \oplus \ell_2 \ .$$

The next result generalizes this to martingale difference sequences, e.g., block bases of (h_i).

Theorem 17. [Bu2], [BDG] *Let $2 < p < \infty$. There exists $C_p < \infty$ so that if (z_i) is a martingale difference sequence in L_p with respect to the sequence of σ-algebras $(\mathcal{F}_n)$, then*

$$\Big\| \sum z_i \Big\|_p \overset{C_p}{\sim} \Big(\sum \|z_i\|_p^p\Big)^{1/p} \vee \Big\| \Big(\sum \mathbb{E}_{\mathcal{F}_i}(z_{i+1}^2)\Big)^{1/2} \Big\|_p \ .$$

Recall something we said earlier. Suppose that $(x_i) \subseteq S_{L_p}$ is weakly null. Passing to a subsequence we obtain (y_i) which, by perturbing, we may assume is a block basis of (h_i). Passing to a further subsequence we may assume $\varepsilon \equiv \lim_i \|y_i\|_2$ exists. If $\varepsilon = 0$ a subsequence of (y_i) is equivalent to the unit vector basis of ℓ_p by the [KP] arguments. Otherwise we have (essentially)

$$\begin{aligned} \varepsilon \Big(\sum |a_i|^2\Big)^{1/2} = \Big\| \sum a_i y_i \Big\|_2 &\le \Big\| \sum a_i y_i \Big\|_p \\ &\le C(p) \Big(\sum |a_i|^2\Big)^{1/2} , \end{aligned}$$

using the fundamental inequality, Proposition 6. Thus $[(y_i)]$ embeds into $\ell_p \oplus \ell_2$ with (y_i) as a block basis of the natural basis for $\ell_p \oplus \ell_2$.

Johnson, Maurey, Schechtman and Tzafriri obtained a stronger version of this dichotomy using Theorem 17.

Theorem 18. [JMST] *Let $2 < p < \infty$. There exists $D_p < \infty$ with the following property. Every normalized weakly null sequence in L_p admits a subsequence (x_i) satisfying, for some $w \in [0,1]$ and all $(a_i) \subseteq \mathbb{R}$,*

$$\Big\| \sum a_i x_i \Big\|_p \overset{D_p}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee w \Big(\sum |a_i|^2 \Big)^{1/2} .$$

We are now ready for an intrinsic characterization of when $X \subseteq L_p$ embeds into $\ell_p \oplus \ell_2$.

Theorem 19. [HOS] *Let $X \subseteq L_p$, $2 < p < \infty$. The following are equivalent.*

a) $X \hookrightarrow \ell_p \oplus \ell_2$

b) *Every weakly null tree in S_X admits a branch (x_i) satisfying for some K and all (a_i)*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i x_i \Big\|_2$$
$$\approx \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum |a_i|^2 \|x_i\|_2^2 \Big)^{1/2} .$$

c) *Every weakly null tree in S_X admits a branch (x_i) satisfying for some K and $(w_i) \subseteq [0,1]$ and all (a_i),*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum |a_i|^2 w_i^2 \Big)^{1/2} .$$

d) *There exists K so that every weakly null sequence in S_X admits a subsequence (x_i) satisfying the condition in b):*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i x_i \Big\|_2$$
$$\approx \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum |a_i|^2 \varepsilon^2 \Big)^{1/2}$$
$$\textit{where } \ \varepsilon = \lim_i \|x_i\|_2 .$$

Condition c) just says that every weakly null tree in S_X admits a branch equivalent to a block basis of the natural basis for $\ell_p \oplus \ell_2$ (discussed more below).

Conditions b) and c) do not require K to be universal but the "all weakly null trees..." hypothesis yields this.

The latter "$\approx$" near equalities in b) (and d)) come from the fact that every weakly null tree in S_{L_p} can be first pruned to a full subtree so that each branch is essentially a normalized block basis of (h_i).

Condition d) is an anomaly in that usually "every sequence has a subsequence..." is a vastly different condition than "every tree admits a branch...". Here the special nature of L_p is playing a role.

The embedding of X into $\ell_p \oplus \ell_2$ will follow the clue from Theorem 15 by producing a blocking (H_n) of (h_i) and embedding X naturally into

$$(\sum H_n)_p \oplus (\sum (H_n, \|\cdot\|_2))_2 \ .$$

Thus if $x = \sum x_n$, $x_n \in H_n$ then $\|x\| \sim (\sum \|x_n\|^p)^{1/p} \vee (\sum \|x_n\|_2^2)^{1/2}$.

The proof of b) $\Rightarrow$ a) is much like that of Theorem 14. We produce a blocking (H_n) of (h_n) so that X naturally embeds into $(\sum H_n)_p \oplus (\sum (H_n, \|\cdot\|_2))_2 \sim \ell_p \oplus \ell_2$. In fact we obtain a more general result.

A basis (v_i) is 1-*subsymmetric* if it is 1-unconditional and $\|\sum a_i v_i\| = \|\sum a_i v_{n_i}\|$ for all (a_i) and all $n_1 < n_2 < \cdots$.

Theorem 20. *Let X and Y be Banach spaces with X reflexive. Let V be a space with a 1-subsymmetric normalized basis (v_i) and let $T : X \to Y$ be a bounded linear operator. Assume that for some C every normalized weakly null tree in X admits a branch (x_n) satisfying:*

$$\Big\|\sum a_n x_n\Big\|_X \overset{C}{\sim} \Big\|\sum a_n v_n\Big\|_V \vee \Big\|T\Big(\sum a_n x_n\Big)\Big\|_Y \ .$$

Then if $X \subseteq Z$, a reflexive space with an FDD(E_i), *there exists a blocking (G_i) of (E_i) so that X naturally embeds into $(\sum G_i)_V \oplus Y$: if $x = \sum x_i$, $x_i \in G_i$ then $x \mapsto (x_i) \oplus Tx \in (\sum G_i)_V \oplus Y$.*

This is applied to $V = \ell_p$, $Z = L_p$ and $Y = L_2$ where $T : X \to L_2$ is the identity map.

So we obtain b) $\Rightarrow$ a) and clearly a) $\Rightarrow$ c). Indeed suppose that $X \subseteq (\ell_p \oplus \ell_2)_\infty$. Then given a weakly null tree in X some branch (x_i) is a perturbation of a normalized block basis (y_i) of the unit vector basis for $\ell_p \oplus \ell_2$. Thus if $\|y_i\|_{\ell_p} = c_i$ and $\|y_i\|_{\ell_2} = w_i$ then $\|\sum a_i y_i\| = (\sum |a_i|^p |c_i|^p)^{1/p} \vee (\sum |a_i|^2 w_i^2)^{1/2}$. From Proposition 6, $\|\sum a_i y_i\|_{(\ell_p \oplus \ell_2)_p} \geq (\sum |a_i|^p)^{1/p}$, hence

$$\begin{aligned}(\sum |a_i|^p)^{1/p} \vee (\sum |a_i|^2 w_i^2)^{1/2} &\leq \|\sum a_i y_i\|_{(\ell_p \oplus \ell_2)_p} \leq 2\|\sum a_i y_i\| \\ &\leq 2\left[(\sum |a_i|^p)^{1/p} \vee (\sum |a_i|^2 w_i^2)^{1/2}\right] \ .\end{aligned}$$

To see c) $\Rightarrow$ b) we begin with a weakly null tree in S_X and choose a branch (x_i) satisfying the c) condition:

$$\Big\|\sum a_i x_i\Big\| \overset{K}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big(\sum |a_i|^2 |w_i|^2\Big)^{1/2} \ .$$

Now we could first have "pruned" our tree so that each branch may be assumed to be a block basis of (h_i), by perturbations. We want to say that for some K',

$$\Big\| \sum a_i x_i \Big\| \overset{K'}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i x_i \Big\|_2 \ .$$

(We have $\overset{K'}{\geq}$ by the fundamental inequality.)

If this fails we can find a block basis (y_n) of (x_n),

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i \ , \ \text{with} \ \sum_{i=k_{n-1}+1}^{k_n} w_i^2 a_i^2 = 1$$

$$\text{and} \ \Big(\sum_{i=k_{n-1}+1}^{k_n} |a_i|^p \Big)^{1/p} \vee \|y_n\|_2 < 2^{-n} \ .$$

But then from the c) condition (y_n) is equivalent to the unit vector basis of ℓ_2 and from the above condition and the [KP] argument, a subsequence is equivalent to the unit vector basis of ℓ_p, a contradiction.

Note that b) $\Rightarrow$ d) since if (x_i) is a normalized weakly null sequence and we define $(x_\alpha)_{\alpha\in T_\infty}$ by $x_{(n_1,\dots,n_k)} = x_{n_k}$ then the branches of $(x_\alpha)_{\alpha\in T_\infty}$ coincide with the subsequences of (x_n). Note that the condition d) just says we may take the weight "w" in [JMST] to be "$\lim_i \|x_i\|_2$".

It remains to show d) $\Rightarrow$ b) in Theorem 19 and this will complete the proof of Theorem 21. The idea is to use Burkholder's inequality using d) on nodes of a weakly null tree, following the scheme of [JMST] to accomplish this. That argument will obtain a branch $(x_n) = (x_{\alpha_n})$, $\alpha_n = (m_1, \dots, m_n)$ with

$$\Big\| \sum a_i x_i \Big\| \sim \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum w_i^2 a_i^2 \Big)^{1/2}$$

where $w_i \overset{C(p)}{\sim} \lim_n \|x_{(\alpha_n, n)}\|_2$ using d). $\square$

Our next goal is to show that if $X \subseteq L_p$ and X dos not embed into $\ell_p \oplus \ell_2$ then X contains an isomorphic copy of $(\sum \ell_2)_p$. The idea will be to use the failure of d) to show $(\sum \ell_2)_p \hookrightarrow X$. In the [KP] argument we obtained a sequence $(x_i) \subseteq S_X$ with the x_i's becoming more and more skinny:

$$\lim_i m[|x_i| \geq \varepsilon] = 0 \ \text{ for all } \ \varepsilon > 0$$

and then extracted an ℓ_p subsequence, of almost disjointly supported functions. Here we want to replace x_i by a sequence of skinny K-isomorphic copies of ℓ_2.

Theorem 21. *Let $X \subseteq L_p$, $2 < p < \infty$. If X does not embed into $\ell_p \oplus \ell_2$ then $(\sum \ell_2)_p \hookrightarrow X$.*

We want to produce $X_i \subseteq X$, $X_i \overset{K}{\sim} \ell_2$ where two things happen. First for all $\varepsilon > 0$ there exists i so that if $x \in S_{X_i}$ then $m[|x| \geq \varepsilon] < \varepsilon$. Secondly we need that X_i is not too skinny, namely each B_{X_i} is p-uniformly integrable.

Definition. $A \subseteq L_p$ is *p-uniformly integrable* if $\forall\ \varepsilon > 0\ \exists\ \delta > 0\ \forall\ m(E) < \delta\ \forall\ z \in A$, we have $\int_E |z|^p < \varepsilon$.

Lemma. *Assume for some K and all n there exists $(x_i^n)_{n=1}^\infty \subseteq S_X$ with $\lim_i \|x_i^n\|_2 = \varepsilon_n \downarrow 0$ and $(x_i^n)_i$ is K-equivalent to the unit vector basis of ℓ_2. Then $(\sum \ell_2)_p \hookrightarrow X$.*

Sketch of proof. Note that if $y = \sum_i a_i x_i^n$ has norm 1 then, assuming as we may that $(x_i^n)_i$ is a block basis of (h_i) and $\|x_i^n\|_2 \approx \varepsilon_n$ then

$$\|y\|_2 \approx \Big(\sum a_i^2 \|x_i^n\|_2^2\Big)^{1/2} \lesssim K\varepsilon_n\ .$$

So we have a sequence of skinny $K - \ell_2$'s inside of X. We would like to have if $y^n \in [(x_i^n)_i]$ then they are essentially disjointly supported so $\|\sum y^n\| \sim (\sum \|y^n\|^p)^{1/p}$, as in the [KP] argument. Unlike in [KP] we cannot select one y_n from each $[(x_i^n)_i]$ and pass to a subsequence. We need to fix a given $[(x_i^n)_i]$ for large n so it is skinny enough based on the earlier selections of subspaces and also so that its unit ball is p-uniformly integrable so that future selections of $[(x_i^m)_i]$ will be essentially disjoint from it.

To achieve this we need a sublemma.

Sublemma. *Let $Y \subseteq L_p$, $2 < p < \infty$, with $Y \sim \ell_2$. There exists $Z \subseteq Y$ with S_Z p-uniformly integrable.*

This is proved in two steps. First showing a normalized martingale difference sequence (x_n) with $\{(x_n)\}$ p-uniformly integrable has $A = \{\sum a_i x_i : \sum a_i^2 \leq 1\}$ also p-uniformly integrable by a stopping time argument.

The general case is to use the subsequence splitting lemma to write a subsequence of an ℓ_2 basis as $x_i = y_i + z_i$ where the (y_i) are a p-uniformly integrable (perturbation of) a martingale difference sequence and the z_i's are disjointly supported and then use an averaging argument to get a block basis where the z_i's disappear. $\square$

The subsequence splitting lemma is a nice exercise in real analysis: Given a bounded $(x_i') \subseteq L_1$ there exists a subsequence $(x_i) \subseteq (x_i')$ with $x_i = y_i + z_i$, $y_i \wedge z_i = 0$, (y_i) is uniformly integrable and the z_i's are disjointly supported.

Now we return to condition d) in Theorem 19 and recall by [JMST] every weakly null sequence in S_X has a subsequence (x_i) with for some $w \in [0,1]$,

$$\left\| \sum a_i x_i \right\| \overset{D_p}{\sim} \left(\sum |a_i|^p \right)^{1/p} \vee w \left(\sum |a_i|^2 \right)^{1/2}$$

and d) asserts that for some absolute C, $w \overset{C}{\sim} \lim_i \|x_i\|_2$. Now clearly we can assume that $w \geq \lim_i \|x_i\|_2$ and if d) fails we can use this to construct our ℓ_2's satisfying the lemma and thus obtain $(\sum \ell_2)_p \hookrightarrow X$.

Indeed d) fails yields that we can take a normalized block basis (y_i) of a given (x_i) failing the condition for a large C to obtain $(y_i) \overset{D_p}{\sim} \ell_2$ basis yet $\|y_i\|_2$ remains small. $\square$

So we have the dichotomy for $X \subseteq L_p$, $2 < p < \infty$. Either

- $X \hookrightarrow \ell_p \oplus \ell_2$ or
- $(\sum \ell_2)_p \hookrightarrow X$.

In the latter case using L_p is stable we can get for all $\varepsilon > 0$, $(\sum \ell_2)_p \overset{1+\varepsilon}{\hookrightarrow} X$.

The theory of stable spaces was developed by Krivine and Maurey [KM]. X is *stable* if for all bounded $(x_n), (y_n) \subseteq X$,

$$\lim_m \lim_n \|x_n + y_m\| = \lim_n \lim_m \|x_n + y_m\|$$

provided both limits exist. They proved that if X is stable then for some p and all $\varepsilon > 0$, $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$. They also proved L_p is stable, $1 \leq p < \infty$.

We have obtained in our proof that if $X \not\hookrightarrow \ell_p \oplus \ell_2$ then for some K and all $\varepsilon > 0$ there exist $X_n \subseteq X$, $X_n \overset{K}{\sim} \ell_2$ and if $x_n \in X_n$, $\| \sum x_n \| \overset{1+\varepsilon}{\sim} (\sum \|x_n\|^p)^{1/p}$. Using L_p is stable we can choose $Y_n \subseteq X_n$, $Y_n \overset{1+\varepsilon}{\sim} \ell_2$ for all n.

In fact we can get $(\sum \ell_2)_p$ complemented in X via the next result.

We note first that if $(x_i) \subseteq S_{L_p}$ is K-equivalent to the unit vector basis of ℓ_2 then, as mentioned earlier, by [PR] it is $C(K)$-complemented in L_p by some projection P. Also P must have the form (true for any projection of any space onto ℓ_2)

$Px = \sum x_i^*(x) x_i$ where (x_i^*) is biorthogonal to (x_i) and is weakly null in $L_{p'}$ $(\frac{1}{p} + \frac{1}{p'} = 1)$.

Proposition 22. *For all n let $(y_i^n)_i$ be a normalized basic sequence in L_p, $2 < p < \infty$, which is K-equivalent to the unit vector basis of ℓ_2 and so that for $y_n \in [(y_i^n)_i]$,*

$$\left\| \sum y_n \right\| \overset{K}{\sim} \left(\sum \|y_n\|^p \right)^{1/p} .$$

Then there exists subsequences $(x_i^n)_i \subseteq (y_i^n)_i$, for each n, so that $[\{x_i^n : n, i \in \mathbb{N}\}]$ is complemented in L_p.

Proof. By [PR] each $[(y_i^n)_i]$ is $C(K)$-complemented in L_p via projections $P_n = \sum_m y_m^{n\,*}(x) y_m^n$. Passing to a subsequence and using a diagonal argument and perturbing we may assume there exists a blocking (H_m^n) of (h_i), in some order over all n, m, so that for all n, m, $\text{supp}(y_m^{n\,*})$, $\text{supp}(y_m^n) \subseteq H_m^n$. This uses $y_m^n \xrightarrow{w} 0$ and $y_m^{n\,*} \xrightarrow{w} 0$ (in $L_{p'}$) as $m \to \infty$ for each n. Set $Py = \sum_{n,m} y_m^{n\,*}(y) y_m^n$. We show P is bounded, hence a projection onto a copy of $(\sum \ell_2)_p$.

Let $y = \sum_{n,m} y(n,m)$, $y(n,m) \in H_n^m$.

$$\begin{aligned}\|Py\| &= \Big\| \sum_n \sum_m y_m^{n\,*}(y(n,m)) y_m^n \Big\| \\ &\sim \Big(\sum_n \Big(\sum_m |y_m^{n\,*}(y(n,m))|^2 \Big)^{p/2} \Big)^{1/p} .\end{aligned}$$

Now

$$\Big(\sum_m |y_m^{n\,*}(y(n,m))|^2 \Big)^{1/2} \sim \|P_n y(n)\| \le C(K) \|y(n)\|$$

where $y(n) = \sum_m y(n,m)$. So

$$\|Py\| \le \bar{C}(K) \Big(\sum \|y_n\|^p \Big)^{1/p} \le \bar{\bar{C}}(K) \|y\| .$$

$\square$

Remarks. The proof of Proposition 22 above is due to Schechtman. He also proved by a different much more complicated argument that the proposition extends to
$1 < p < 2$.

In [HOS] the proofs of all the results are also considered using Aldous' [Ald] theory of random measures. We are able to show if $(\sum \ell_2)_p \hookrightarrow X \subseteq L_p$, $2 < p < \infty$, then given $\varepsilon > 0$ there exists $(\sum Y_n)_p \overset{1+\varepsilon}{\hookrightarrow} X$, $d(Y_n, \ell_2) < 1 + \varepsilon$ and moreover: there exist disjoint sets $A_n \subseteq [0,1]$ with for all n, $y \in Y_n$, $\|y|_{A_n}\| \ge (1 - \varepsilon 2^{-n}) \|y\|$ and $[Y_n : n \in \mathbb{N}]$ is $(1+\varepsilon)$ C_p^{-1} complemented in L_p where C_p is the norm of a symmetric normalized Gaussian random variable in L_p. This is best possible by [GLR].

We can also deduce the [JO2] result: $X \subseteq L_p$, $2 < p < \infty$, and X is a quotient of a subspace of $\ell_p \oplus \ell_2 \Rightarrow X \hookrightarrow \ell_p \oplus \ell_2$, by showing that such an X cannot contain $(\sum \ell_2)_p$.

We shall prove something more general, namely that $(\sum \ell_q)_p$ is not a quotient of a subspace of $\ell_p \oplus \ell_q$ when $p, q > 1$ and $p \ne q$. By duality it will be enough to consider the case $p > q$. For elements $w = (w_1, w_2)$ of $\ell_p \oplus \ell_q$ we shall write $\|w\|_p = \|w_1\|_p$, $\|w\|_q = \|w_2\|_q$ and $\|w\| = \|w\|_p \vee \|w\|_q$.

Lemma. *Let $1 < q < p < \infty$ and let W be a subspace of $\ell_p \oplus \ell_q$. Let $X = \ell_q$, let $Q : W \to X$ be a quotient mapping and let λ be a constant with $0 < \lambda < \|Q\|^{-1}$. For every $M > 0$ there is a finite co-dimensional subspace Y of X such that, for $w \in W$ we have*

$$\|w\| \leq M,\ Q(w) \in Y,\ \|Q(w)\| = 1 \implies \|w\|_q > \lambda.$$

Proof. Suppose otherwise. We can find a normalized block basis (x_n) in X and elements w_n of W with $\|w_n\| \leq M$, $Q(w_n) = x_n$ and $\|w_n\|_q \leq \lambda$. Taking a subsequence and perturbing slightly, we may suppose that $w_n = w + w_n'$, where (w_n') is a block basis in $\ell_p \oplus \ell_q$, satisfying $\|w_n'\| \leq M$, $\|w_n'\|_q \leq \lambda$.

Since $Q(w) = \text{w-lim}\, Q(w_n) = 0$, we see that $Q(w_n') = x_n$. We may now estimate as follows using the fact that the w_n' are disjointly supported:

$$\Big\| \sum_{n=1}^N w_n' \Big\| = \Big(\sum_{n=1}^N \|w_n'\|_p^p \Big)^{1/p} \vee \Big(\sum_{n=1}^N \|w_n'\|_q^q \Big)^{1/q} \leq N^{1/p} M \vee N^{1/q} \lambda.$$

Since the x_n are normalized blocks in $X = \ell_q$ we have

$$N^{1/q} = \Big\| \sum_{n=1}^N x_n \Big\| \leq \|Q\| \, \Big\| \sum_{n=1}^N w_n' \Big\| \leq M\|Q\|N^{1/p} \vee \lambda\|Q\|N^{1/q}.$$

Since $\lambda\|Q\| < 1$, this is impossible once N is large enough. □

Proposition 23. *If $1 < q < p < \infty$ then $(\sum \ell_q)_p$ is not a quotient of a subspace of $\ell_p \oplus \ell_q$.*

Proof. Suppose, if possible, that there exists a quotient operator

$$\ell_p \oplus \ell_q \supseteq Z \xrightarrow{\text{Q}} X = \Big(\bigoplus_{n \in \mathbb{N}} X_n \Big)_p$$

where $X_n = \ell_q$ for all n. Let K be a constant such that $T[KB_Z] \supseteq B_X$, let λ be fixed with $0 < \lambda < \|Q\|^{-1}$, choose a natural number m with $m^{1/q - 1/p} > K\lambda^{-1}$, and set $M = 2Km^{1/p}$.

Applying the lemma, we find, for each n, a finite co-dimensional subspace Y_n of X_n such that

$$z \in MB_Z,\ Q(z) \in Y_n,\ \|Q(z\| = 1 \implies \|z\|_q > \lambda. \tag{2}$$

For each n, let $(e_i^{(n)})$ be a sequence in Y_n, 1-equivalent to the unit vector basis of ℓ_q. For each m-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in \mathbb{N}^m$, let $z(\mathbf{i}) \in Z$ be chosen with

$$Q(z(\mathbf{i}) = e_{i_1}^{(1)} + e_{i_2}^{(2)} + \cdots + e_{i_m}^{(m)},$$

and $\|z(\mathbf{i})\| \leq Km^{1/p}$.

Taking subsequences in each co-ordinate, we may suppose that the following weak limits exist in Z

$$\begin{aligned} z(i_1, i_2, \ldots, i_{m-1}) &= \text{w-lim}_{i_m \to \infty} z(i_1, i_2, \ldots, i_m) \\ &\vdots \\ z(i_1, i_2, \ldots, i_j) &= \text{w-lim}_{i_{j+1} \to \infty} z(i_1, i_2, \ldots, i_{j+1}) \\ &\vdots \\ z(i_1) &= \text{w-lim}_{i_2 \to \infty} z(i_1, i_2). \end{aligned}$$

Notice that, for all j and all $i_1, i_2, \ldots, i_j$, the following hold:

$$\begin{aligned} Q(z(i_1, \ldots, i_j) &= e^{(1)}_{i_1} + \cdots + e^{(j)}_{i_j} \\ \|z(i_1, \ldots, i_j)\| &\leq Km^{1/p} \\ \|z(i_1, \ldots, i_j) - z(i_1, \ldots, i_{j-1})\| &\leq 2Km^{1/p} = M. \end{aligned}$$

Since $Q(z(i_1, \ldots, i_j) - z(i_1, \ldots, i_{j-1})) = e^{(j)}_{i_j} \in S_{Y_j}$ it must be that

$$\|z(i_1, \ldots, i_j) - z(i_1, \ldots, i_{j-1})\|_q > \lambda, \qquad \text{[by (2)]}. \tag{3}$$

We shall now choose recursively some special i_j in such a way that $\|z(i_1, \ldots, i_j)\|_q > \lambda j^{1/q}$ for all j. Start with $i_1 = 1$; since $\|z(i_1)\| \leq M$ and $Q(z(i_1)) = e^{(1)}_{i_1}$ we certainly have $\|z(i_1)\|_q > \lambda$ by 2. Since $z(i_1, k) - z(i_1) \to 0$ weakly we can choose i_2 such that $z(i_1, i_2) - z(i_1)$ is essentially disjoint from $z(i_1)$. More precisely, because of 3, we can ensure that

$$\|z(i_1, i_2)\|_q = \|z(i_1) + (z(i_1, i_2) - z(i_1))\|_q > (\lambda^q + \lambda^q)^{1/q} = \lambda 2^{1/q}.$$

Continuing in this way, we can indeed choose $i_3, \ldots, i_m$ in such a way that

$$\|z(i_1, \ldots, i_j)\|_q \geq \lambda j^{1/q}.$$

However, for $j = m$ this yields

$$\lambda m^{1/q} \leq Km^{1/p},$$

contradicting our initial choice of m. □

Remark. The proof we have just given actually establishes the following quantitative result: if Y is a quotient of a subspace of $\ell_p \oplus \ell_q$ then the Banach-Mazur distance $d\big(Y, \big(\bigoplus_{j=1}^m \ell_q\big)_p\big)$ is at least $m^{|1/q - 1/p|}$.

We can also obtain some asymptotic results. First we recall the relevant definitions

$$\operatorname{cof}(X) = \{Y \subseteq X : Y \text{ is of finite co-dimension in } X\}\ .$$

Definition. [MMT] Let $(e_i)_1^n$ be a normalized monotone basis. $(e_i) \in \{X\}_n$, the n^{th} asymptotic structure of X, if the following holds;

$$\begin{aligned}
\forall\, \varepsilon > 0\ \forall\ X_1 \in \operatorname{cof}(X)\ \exists\ x_1 \in S_{X_1} \\
\forall\ X_2 \in \operatorname{cof}(X)\ \exists\ x_2 \in S_{X_2} \\
\dots \qquad\qquad\qquad \\
\forall\ X_n \in \operatorname{cof}(X)\ \exists\ x_n \in S_{X_n} \\
\text{with } d_b((x_i)_1^n, (e_i)_1^n) < 1+\varepsilon
\end{aligned}$$

The latter means that for some $AB < 1+\varepsilon$ for all $(a_i)_1^n \subseteq \mathbb{R}$,

$$A^{-1}\Big\|\sum_1^n a_i e_i\Big\| \le \Big\|\sum_1^n a_i x_i\Big\| \le B\Big\|\sum_1^n a_i e_i\Big\| ,$$

i.e., $(x_i)_1^n \overset{1+\varepsilon}{\sim} (e_i)_1^n$. $d_b(\cdot)$ is the basis distance and is defined to be the minimum of such AB.

An alternate way of looking at this when X^* is separable is that $\{X\}_n$ is the smallest closed subset of $(\mathcal{M}_n, d_b(\cdot,\cdot))$ satisfying: $\forall\,\varepsilon > 0$ every weakly null tree (of length n) in S_X admits a branch $(x_i)_1^n$ with $d_b((x_i)_1^n, \{X\}_n) < 1+\varepsilon$. Here $\mathcal{M}_n$ is the set of normalized bases of length n. The metric on $\mathcal{M}_n$ is actually $\log d_b(\cdot,\cdot)$ and $\mathcal{M}_n$ is compact under this metric.

Definition. X is K-*asymptotic* ℓ_p if for all n and all $(e_i)_1^n \in \{X\}_n$, $(e_i)_1^n$ is K-equivalent to the unit vector basis of ℓ_p^n.

The [KP], [JO1] results yield for $X \subseteq L_p$, $2 < p < \infty$

- X is asymptotic $\ell_p \Rightarrow X \hookrightarrow \ell_p$ (since $\ell_2 \not\hookrightarrow X$)
- X is asymptotic $\ell_2 \Rightarrow X \hookrightarrow \ell_2$ (since $\ell_p \not\hookrightarrow X$).

Definition. X is *asymptotically* $\ell_p \oplus \ell_2$ if $\exists\ K\ \forall\ n \forall\ (e_i)_1^n \in \{X\}_n\ \exists\ (w_i)_1^n$ with

$$\Big\|\sum_1^n a_i e_i\Big\| \overset{K}{\sim} \Big(\sum_1^n |a_i|^p\Big)^{1/p} \vee \Big(\sum_1^n w_i^2 a_i^2\Big)^{1/2} .$$

This just says that for some K every weakly null tree of n-levels in S_X admits a branch K-equivalent to a normalized block basis of $\ell_p \oplus \ell_2$.

Proposition 24. *Let $X \subseteq L_p$, $2 < p < \infty$. X is asymptotically $\ell_p \oplus \ell_2$ iff $X \hookrightarrow \ell_p \oplus \ell_2$.*

This follows easily from our results by showing that $(\sum \ell_2)_p$ is not asymptotically $\ell_p \oplus \ell_2$.

Problem. Let $X \subseteq L_p$, $p > 2$. Give an intrinsic characterization of when $X \hookrightarrow (\sum \ell_2)_p$.

In light of the [JO2] $\ell_p \oplus \ell_2$ quotient result (see paragraph 7.1 above) we ask the following.

Problem 25. Let $X \subseteq L_p$ $(2 < p < \infty)$. If X is a quotient of $(\sum \ell_2)_p$ does X embed into $(\sum \ell_2)_p$?

Extensive study has been made of the $\mathcal{L}_p$ spaces, i.e., the complemented subspaces of L_p which are not isomorphic to ℓ_2 (see e.g., [LP] and [LR]). In particular there are uncountably many such spaces [BRS] and even infinitely many which embed into $(\sum \ell_2)_p$ [S1]. Thus it seems that a deeper study of the index in [BRS] will be needed for further progress. However some things, which we now recall, are known.

Theorem 26. [P] *If Y is complemented in ℓ_p then Y is isomorphic to ℓ_p (Proposition 10).*

Theorem 27. [JZ] *If Y is a $\mathcal{L}_p$ subspace of ℓ_p then Y is isomorphic to ℓ_p.*

Theorem 28. [EW] *If Y is complemented in $\ell_p \oplus \ell_2$ then Y is isomorphic to ℓ_p, ℓ_2 or $\ell_p \oplus \ell_2$.*

Theorem 29. [O] *If Y is complemented in $(\sum \ell_2)_p$ then Y is isomorphic to ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$ or $(\sum \ell_2)_p$.*

We recall that X_p is the $\mathcal{L}_p$ discovered by H. Rosenthal [R]. For $p > 2$, X_p may be defined to be the subspace of $\ell_p \oplus \ell_2$ spanned by $(e_i + w_i f_i)$, where (e_i) and (f_i) are the unit vector bases of ℓ_p and ℓ_2, respectively, and where $w_i \to 0$ with $\sum w_i^{2p/p-2} = \infty$. Since $\ell_p \oplus \ell_2$ embeds into X_p, the subspaces of X_p and of $\ell_p \oplus \ell_2$ are (up to isomorphism) the same. For $1 < p < 2$ the space X_p is defined to be the dual of $X_{p'}$ where $1/p + 1/p' = 1$. When restricted to $\mathcal{L}_p$-spaces, the results of this paper lead to a dichotomy valid for $1 < p < \infty$.

Proposition 30. *Let Y be a $\mathcal{L}_p$-space $(1 < p < \infty)$. Either Y is isomorphic to a complemented subspace of X_p or Y has a complemented subspace isomorphic to $(\sum \ell_2)_p$.*

Proof. For $p > 2$ it is shown in [JO2] that a $\mathcal{L}_p$-space which embeds in $\ell_p \oplus \ell_2$ embeds complementedly in X_p. Combining this with the main theorem of the present paper gives what we want for $p > 2$. When $1 < p < 2$, the space X_p is defined to be the dual of $X_{p'}$ and so a simple duality argument extends the result to the full range $1 < p < \infty$. □

It remains a challenging problem to understand more deeply the structure of the $\mathcal{L}_p$-subspaces of X_p and $\ell_p \oplus \ell_2$.

Theorem 31. [JO2] *If Y is a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ (or X_p), $2 < p < \infty$, and Y has an unconditional basis then Y is isomorphic to ℓ_p, $\ell_p \oplus \ell_2$ or X_p.*

It is known [JRZ] that every $\mathcal{L}_p$ space has a basis but it remains open if it has an unconditional basis.

Theorem 32. [JO2] *If Y is a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ $(1 < p < 2)$ with an unconditional basis then Y is isomorphic to ℓ_p or $\ell_p \oplus \ell_2$.*

So the main open problem for small $\mathcal{L}_p$ spaces is to overcome the unconditional basis requirement of 31 and 32.

Problem 33. (a) Let X be a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ $(2 < p < \infty)$. Is X isomorphic to ℓ_p, $\ell_p \oplus \ell_2$ or X_p?

(b) Let X be a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ $(1 < p < \infty)$. Is X isomorphic to ℓ_p or $\ell_p \oplus \ell_2$?

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