A characterization of weight-regular partitions of graphs

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Abstract

A partition $\mathcal{P} = \{V_1, \ldots, V_m\}$ of the vertex set V of a graph is regular if, for all i, j, the number of neighbors which a vertex in V_i has in the set V_j is independent of the choice of vertex in V_i. The natural generalization of a regular partition, which makes sense also for non-regular graphs, is the so-called weight-regular partition, which gives to each vertex $u \in V$ a weight which equals the corresponding entry ν_u of the Perron eigenvector $\boldsymbol{\nu}$. In this work we investigate when a weight-regular partition of a graph is regular in terms of double stochastic matrices. Inspired by a characterization of regular graphs by Hoffman, we provide a new characterization of weight-regular partitions by using a Hoffman-like polynomial.

Keywords: weight-regular partition, regular partition, double stochastic matrix, polynomial

1 Introduction

Let G be a connected graph with vertex set V, adjacency matrix $\boldsymbol{A}$, positive eigenvector $\boldsymbol{\nu}$ and corresponding eigenvalue λ_1. A *regular* or *equitable partition* of the matrix $\boldsymbol{A}$ is a partition of the vertex set into parts V_i such that

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each vertex in V_i has the same number b_{ij} of neighbours in part V_j, for any j. Regular partitions have been widely studied in the literature and provide a handy tool for obtaining inequalities and regularity results concerning the structure of regular graphs. In particular, characterizations of regular partitions and its application to obtain tight bounds for several graph parameters have been previously obtained [5].

The natural generalization of a regular partition, which makes sense also for non-regular graphs, is the so-called *weight-regular partition*. Its definition is based on giving to each vertex $u \in V$ a weight which equals the corresponding entry ν_u of $\boldsymbol{\nu}$. Such weights "regularize" the graph, leading to a kind of regular partition that can be useful for general graphs. Weight partitions have been shown to be a powerful tool used to extend several relevant results for non-regular graphs. Weight partitions were first used by Haemers in 1970 [2] (see Theorem 6) to provide an alternative proof of Hoffman inequality [7] for the chromatic number of a general graph. In [4] and [3], Fiol and Garriga defined them formally and used them to obtain several bounds for parameters of non-regular graphs. Examples of such results are an extension of Hoffman's bound for the chromatic number or a generalization of the Lovász bound for the Shannon capacity of a graph. Moreover, weight-regular partitions have been used to show that an upper bound for the weight-independence number is best possible [4] and to obtain spectral characterizations of distance-regularity around a set and spectral characterizations of completely regular codes [3].

Note that regular partitions are obviously weight-regular. The converse, however, is not true in general. In this work we investigate when a weight-regular partition is regular in terms of double stochastic matrices. The second part of this work is inspired by the article of Hoffman [6] in which he characterizes regular graphs in terms of the Hoffman polynomial. We obtain a new characterization of weight-regular partitions by using a Hoffman-like polynomial. Up until now, the only known characterization of weight-regular partitions appears in [4] (see Lemma 2.2 and Lemma 2.3). A potential application of weight-regular partitions is to provide new families of graphs that attain equality in known bounds for general graphs, as the Hoffman's spectral lower bound for the chromatic number of a graph [6], which is one of the best known results in spectral graph theory.

This article is organized as follows. Section 2 recalls some definitions and terminologies about weight partitions. In section 3.1 we characterize when weight-regular partitions are regular in terms of double stochastic matrices. Finally, in section 3.2 we give a characterization of weight-regular partitions by using polynomials.

2 Preliminaries

In this section we introduce some definitions and properties relating to weight partitions. The all-ones matrix is denoted $\mathbf{J}$, and $\mathbf{1}$ is the all-ones vector.

Let $G = (V, E)$ be a simple and connected graph on $n = |V|$ vertices, with adjacency matrix $\boldsymbol{A}$, eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and spectrum

$$\mathrm{sp}G = \mathrm{sp}\boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\},$$

where the different eigenvalues of G are in decreasing order $\theta_0^{m_0} > \theta_1^{m_1} > \cdots > \theta_d^{m_d}$, and the superscripts stand for their multiplicities $m_i = m(\theta_i)$. Since G is connected (so $\boldsymbol{A}$ is irreducible), Perron-Frobenius Theorem assures that λ_1 is simple, positive and has positive eigenvector. If G is non-connected, the existence of such an eigenvector is not guaranteed, unless all its connected components have the same maximum eigenvalue. Throughout this work, the positive eigenvector associated with the largest (positive and with multiplicity one) eigenvalue λ_1 is denoted by $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)^\top$. This eigenvector is normalized in such a way that its minimum entry (in each connected component of G) is 1. For instance, if G is regular, we have $\boldsymbol{\nu} = \mathbf{1}$.

Let $\mathcal{P}$ be a partition of the vertex set $V = V_1 \cup \cdots \cup V_m$, $1 \leq m \leq n$. Consider the map $\boldsymbol{\rho} : V \longrightarrow \mathbb{R}^+$ defined by $\boldsymbol{\rho}U := \sum_{u \in U} \rho_u \boldsymbol{e}_u$. In particular, for weight-partitions we consider the map $\boldsymbol{\rho} : \mathcal{P}(V) \longrightarrow \mathbb{R}^n$ defined by

$$\boldsymbol{\rho}U := \sum_{u \in U} \nu_u \boldsymbol{e}_u$$

for any $U \neq \emptyset$, where $\boldsymbol{e}_u$ represents the u-th canonical (column) vector, $\boldsymbol{\rho}\emptyset = \mathbf{0}$ and ν is the eigenvector of the largest eigenvalue. Note that, with $\boldsymbol{\rho}u := \boldsymbol{\rho}\{u\}$, we have $||\boldsymbol{\rho}u|| = \nu_u$, so that we can see $\boldsymbol{\rho}$ as a function which assigns weights to the vertices of G. In doing so we "regularize" the graph, in the sense that the *weight-degree* of each vertex $u \in V$ becomes a constant:

$$\delta_u^* := \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \lambda_1 \tag{1}$$

Given $\mathcal{P} = \{V_1, \ldots, V_m\}$, for $u \in V_i$ we define the *weight-intersection numbers* as follows:

$$b_{ij}^*(u) := \frac{1}{\nu_u} \sum_{v \in G(u) \cap V_j} \nu_v \qquad (1 \leq i, j \leq m). \tag{2}$$

Observe that the sum of the weight-intersection numbers for all $1 \le j \le m$ gives the weight-degree of each vertex $u \in V_i$:

$$\sum_{j=1}^{m} b^*_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in G(u)} \nu_v = \delta^*_u = \lambda_1.$$

A matrix characterization of weight partitions can be done via the following matrix associated with any partition $\mathcal{P}$. The *normalized weight-characteristic matrix* of $\mathcal{P}$ is the $n \times m$ matrix $\overline{\mathbf{S}}^* = (\overline{s}^*_{uj})$ with entries

$$\overline{s}^*_{uj} = \begin{cases} \frac{\nu_u}{||\boldsymbol{\rho} V_j||} & \text{if } u \in V_j, \\ 0 & \text{otherwise.} \end{cases}$$

that satisfies $(\overline{\mathbf{S}}^*)^\top \overline{\mathbf{S}}^* = \boldsymbol{I}$. We define the *normalized weight-quotient matrix* of $\boldsymbol{A}$ with respect to $\mathcal{P}$, $\overline{\boldsymbol{B}}^* = (\overline{b}^*_{ij})$, as

$$\overline{\boldsymbol{B}}^* = (\overline{\mathbf{S}}^*)^\top \boldsymbol{A} \overline{\mathbf{S}}^* = \boldsymbol{D}^{-1} (\widetilde{\mathbf{S}}^*)^\top \boldsymbol{A} \widetilde{\mathbf{S}}^* \boldsymbol{D}^{-1} = \boldsymbol{D}^{-1} \widetilde{\boldsymbol{B}}^* \boldsymbol{D}^{-1},$$

and hence $\overline{b}^*_{ij} = \frac{\widetilde{b}^*_{ij}}{||\boldsymbol{\rho} V_i|| \, ||\boldsymbol{\rho} V_j||}$.

A partition $\mathcal{P}$ is called *weight-regular* whenever the weight-intersection numbers do not depend on the chosen vertex $u \in V_i$, but only on the subsets V_i and V_j. In such a case, we denote them by

$$b^*_{ij}(u) = b^*_{ij} \qquad \forall u \in V_i$$

and we consider the $m \times m$ matrix $\boldsymbol{B}^* = (b^*_{ij})$, called the *weight-regular-quotient matrix* of $\boldsymbol{A}$ with respect to $\mathcal{P}$.

The following result was partially stated in [3].

Lemma 2.1 *A $\mathcal{P} = \{V_1, V_2, \ldots, V_m\}$ partition of a graph G is regular if and only if it is weight-regular and the map on V, denoted $\boldsymbol{\rho} : u \to \nu_u$, is constant over each V_k, say ν_k. Then, it holds that the quotient matrix entries of the regular partition (b_{ij}) and the quotient matrix entries of the weight-regular partition (b^*_{ij}) satisfy*

$$b^*_{ij} = \frac{\nu_j}{\nu_i} b_{ij}.$$

In [3] an example of a weight-regular partition which is not regular is given. Many other examples arise, for instance, from the bipartition of any connected bipartite graph, which is always weight-regular but does not always define a regular partition.

3 Main results

Our main results are the two following characterizations of weight-regularity.

3.1 Double stochastic matrices and weight-regularity

As mentioned above, weight-regular partitions are not necessarily regular. In this section we give a characterization of weight-regular partitions being regular in terms of double stochastic matrices. A matrix is *double stochastic* if it is nonnegative and each of its rows and each of its columns sums up to one. If $\boldsymbol{A}$ is the adjacency matrix of a graph G,we denote by $\Omega(\boldsymbol{A})$ the set of all double stochastic matrices which commute with $\boldsymbol{A}$. Note that $\Omega(\boldsymbol{A})$ is a convex polytope since it consists of all matrices $\mathbf{X}$ such that

$$\mathbf{X}\boldsymbol{A} = \boldsymbol{A}\mathbf{X}, \quad \mathbf{X}\mathbf{1} = \mathbf{1}\mathbf{X}, \quad \mathbf{X} \geq 0.$$

Lemma 3.1 *Let $\boldsymbol{A}$ be the adjacency matrix of a graph G, and let $\mathcal{P}$ be a weight-regular partition of the vertex set with normalized weight-characteristic matrix $\overline{\mathbf{S}}^*$. Then $\mathcal{P}$ is regular if and only if $\boldsymbol{A}$ and $\overline{\mathbf{S}}^*\overline{\mathbf{S}}^{*\top}$ commute.*

The above result yields to the following corollary.

Corollary 3.2 *Let $\mathcal{P}$ be a weight-regular partition of the vertices of G with normalized weight-characteristic matrix $\overline{\mathbf{S}}^*$. Then $\mathcal{P}$ is regular if and only if $\overline{\mathbf{S}}^*\overline{\mathbf{S}}^{*\top} \in \Omega(\boldsymbol{A})$.*

3.2 Polynomials and weight-regularity

In [6], Hoffman proved that a (connected) graph G is regular if and only if $H(\boldsymbol{A}) = \mathbf{J}$, in which case H becomes the Hoffman polynomial. An analogous of Hoffman's result for bipartite graphs was given in [1]. The following result proves a natural extension of Hoffman's result for weight-regular partitions of a graph and it leads to generalizations of the characterizations for regular and biregular graphs in [6] and [1], respectively. Recall that b^*_{ij} denote the entries of the weight-quotient matrix defined in Section 2.

Theorem 3.3 *Let G be a connected graph with a partition of its vertices into m sets, $\mathcal{P} = \{V_1, \ldots, V_m\}$, such that $n = n_1 + \cdots + n_m$ and such that the map on V, denoted by $\boldsymbol{\rho} : u \to \nu_u$, is constant over each V_k (say ν_k). Then there* *exists a polynomial* $H \in \mathbb{R}_d[x]$ *such that*

$$H(\boldsymbol{A}) = \begin{pmatrix} b^*_{11}\mathbf{J} & b^*_{12}\mathbf{J} & \cdots & b^*_{1m}\mathbf{J} \\ b^*_{21}\mathbf{J} & b^*_{22}\mathbf{J} & \cdots & b^*_{2m}\mathbf{J} \\ \vdots & & \ddots & \\ b^*_{m1}\mathbf{J} & b^*_{m2}\mathbf{J} & \cdots & b^*_{mm}\mathbf{J} \end{pmatrix} \tag{3}$$

if and only if $\mathcal{P}$ *is a weight-regular partition of* G.

Observe that weight-partitions maintain the structure of the Perron eigenvector $\boldsymbol{\nu}$. As a corollary of Theorem 3.3 we obtain Hoffman's result (take $m = n$ and recall that for a regular graph $\boldsymbol{\nu} = \mathbf{1}$).

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