2-closed abelian permutation groups [⋆]

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Abstract

In this paper we demonstrate that the result by Zelikovskij concerning Königs problem for abelian permutation groups, reported in a recent survey, is false. We propose in this place two results on 2-closed abelian permutation groups which concern the same topic in a more general setting.

Keywords: graph, colored graph, automorphism group, abelian group.

In 1936, König [10] posed the following question: "When can a given abstract group be interpreted as the automorphism group of a graph?". This question was quickly answered by Frucht in 1938 [4], who proved that every abstract group is isomorphic to the automorphism group of a graph. The next natural question in this direction was which permutation groups were (identical with) the automorphishm groups of graphs. This question is known as the *concrete version of König's problem* (see. [14,1,13]).

This second question turned out to be much harder. There exist permutation groups that are not the automorphism groups of any graph. For

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example, the alternating groups or the groups C_n generated by a cyclic permutation of order n are not such groups, while the symmetric group S_n is the automorphism group of the complete graph, and the dihedral group D_n is the automorphism group of the graph being the cycle on n vertices.

So far, the concrete version of König's problem has been solved for the class of *regular* permutation groups. This special class has been studied within the so called *Graphical Regular Representation* problem. A number of partial results has been concluded by Godsil in 1978 with the full characterization presented in [5]. L. Babai [2] has applied this result to obtain a similar characterization of those regular permutation groups that happen to be the automorphism groups of directed graphs.

Another class of permutation groups which seemed relatively easy from the point of view of the problem in question are *cylic* permutation groups, that is, those generated by a single permutation. In [6], Grech corrected and completed earlier results in [11,12] on the cyclic permutation groups whose order is a power of a prime. In [7], we have extended the result to the cyclic permutation groups of an arbitrary order. More about other detailed results and related research the reader may find in surveys [9,13].

In studying the concrete version of König's problem it turns out that the corresponding results for edge-colored graphs have usually simpler and more natural formulation than their counterparts for simple graphs. This has been realized already in H. Wielandt in [14], where the permutation groups that are automorphism groups of colored graphs and digraphs were called 2^*-closed and 2-closed, respectively. (The definitions in [14] are in terms of the invariance groups of systems of binary relations. We note also that, e.g. in [3], the latter term is reserved for the first defined class, but we follow here Wielandt's original terminology). For example, the main result in [7] states that a cyclic permutation group A is 2^*-closed if and only if for every orbit O of A with cardinality $|O| > 2$ there exist another orbit O' of A such that $\gcd(|O|, |O'|) > 2$. The analogous characterization for the cyclic permutation groups that are automorphism groups of simple graphs is much more complicated with many technical details.

The most advanced result in the area is described in [13]: "In 1989, a paper by Zelikovskij [15] appeared in Russian. We are unaware of a translation, so can only report the main result as stated in the English summary: that for every finite abelian permutation group G whose order is relatively prime to 30, the paper provides necessary and sufficient conditions for the existence of a simple graph whose automorphism group is isomorphic to G."

The aim of this paper is to report the results contained in [15] and demonstrate that they are, unfortunately, wrong. We also report our own results on 2-closed abelian permutation groups.

1 Zelikovskij results

Let A be an abelian permutation group, whose orbits are X_i, $i = 1, \dots, n$. By A_i we denote the restriction of A to the orbit X_i (the constituent of A on X_i), by A_i^j we denote the pointwise stabilizer of X_j restricted to the orbit X_i, and by B_i we denote the pointwise stabilizer of the set $X \setminus X_i$ restricted to X_i.

It is not difficult to prove that if A is the automorphism group of a graph then $\bigcap_{j=1, j\neq i}^{n} A_i^j = B_i$, for each $i = 1, \dots, n$. Also, if $|X_i| > 1$, then $A_i \neq B_i$, which follows from a result in Imrich [8]. Zelikovskij claims that these two conditions are sufficient for an abelian group A to be the automorphism group of a graph, provided the order of A is not divisible by $2, 3$ or 5.

Theorem 1.1 (Zelikovskij [15]) *Let A be an abelian group, whose order is not divisible by $2, 3, 5$. Then, A is the automorphism group of a graph if and only if for each nontrivial orbit X_i of A the following conditions hold:*

(i) $\bigcap_{j=1, j\neq i}^{n} A_i^j = B_i$,

(ii) $A_i \neq B_i$.

The proof is based on a lemma that utilizes the following notion of "closure", which we call 2-orbit-closure. Let G be a permutation group, whose orbits are X_i, $i = 1, \dots, n$. A permutation σ that preserves each orbit X_i is called 2-*orbit-compatible* with G, if for each pair of orbits X_i and X_j, $i \neq j$, the restriction of σ to $X_i \cup X_j$ belongs to the restriction of G to $X_i \cup X_j$. The group G is 2-*orbit-closed* if every permutation that is 2-orbit-compatibile with G belongs to G. (The definition in [15] is given in different terms, but the equivalence of both the definitions is straightforward.) One may check, that 2-closure implies 2-orbit-closure.

Lemma 1.2 (Zelikovskij [15]) *If an abelian group A satisfies the condition* (i) *of Theorem 1.1, then A is 2-orbit-closed.*

The proof of this lemma in [15] (as well as the proof in the author's PhD thesis) contains a gap. Rather then reproducing the proof here and pointing out the gap, we construct an example showing that both the lemma and the theorem are false.

Let X be the union of sets $X_i = \{x_0^i, x_1^i, \dots, x_6^i\}$, $i = 1, 2, \dots, 6$. Let A be a permutation group on X consisting of all permutations σ of the following form (for each $k = 0, \dots, 6$):

$\sigma(x_k^1) = x_{k+r}^1$, $\sigma(x_k^2) = x_{k+s}^2$, $\sigma(x_k^3) = x_{k+t}^3$,
$\sigma(x_k^4) = x_{k+r}^4$, $\sigma(x_k^5) = x_{k+s}^5$, $\sigma(x_k^6) = x_{k+t}^6$,
where $r + s + t \equiv 0 \pmod 7$, and all the indices are taken modulo 7. This set of permutations is clearly closed on composition, and therefore it forms a permutation group of order 49.

Note that $X_1, \ldots, X_6$ are orbits of A, and each permutation in A acts in a parallel way on orbits X_1 and X_4, X_2 and X_5, and X_3 and X_6, respectively. In particular, the action of A on $X_1 \cup X_2 \cup X_3$ determines uniquely the (parallel) action of A on $X_4 \cup X_5 \cup X_6$. It is also easily seen that A is abelian.

Now, clearly, A_i is the cyclic group generated by the cyclic permutation $(x_0^i, x_1^i, \ldots, x_6^i)$. For $i = 1, 2, 3$, A_i^{i+3} and A_{i+3}^i are all trivial (i.e., consisting of the identity only), and $A_i^j = A_i$, otherwise. Also B_i is trivial for all $i = 1, \ldots, 6$. It follows that both conditions (i) and (ii) of Theorem 1.1 are satisfied.

We show that A is not the automorphism group for any graph. (or any edge-colored graph). Suppose that there is an (edge-colored) graph Γ on the set X of vertices such that the automorphism group $Aut(\Gamma) = A$. We have in A a permutation σ satisfying (for all k) $\sigma(x_k^1) = x_k^1$ (identity action) and $\sigma(x_k^2) = x_{k+r}^2$ (for arbitrary r). If $\sigma \in Aut(\Gamma)$, then all edges from x_0^1 to X_2 in Γ are of the same color. A converse argument, for x_0^2 and X_1, leads to the conclusion that all edges between X_1 and X_2 are of the same color. Similarly, all edges between X_i and X_j are of the same color for any $i \neq j$ in $\{1, 2, 3\}$. This means, that with no regard to remaining edges, if permutations σ_i on X_i preserve the edges in X_i, respectively, then the permutation $\sigma_1\sigma_2\sigma_3\tau_1\tau_2\tau_3$ is the automorphism of Γ for some suitable permutations τ_1, τ_2, τ_3 on the orbits X_4, X_5, X_6, respectively. This shows that $Aut(\Gamma) \neq A$, a contradiction. (Similar argument works for digraphs).

We show that A is not 2-orbit-closed, neither. Let $A_{i,j}$ denote the restriction of A to $X_i \cup X_j$. We note that $A_{i,i+3}$, for $i = 1, 2, 3$, is the group generated by the permutation $(x_0^i, x_1^i, \ldots, x_6^i)(x_0^j, x_1^j, \ldots, x_6^j)$, which is the so called parallel product of two cyclic groups. Otherwise, $A_{i,j}$ is the group generated by two permutations $(x_0^i, x_1^i, \ldots, x_6^i)$ and $(x_0^j, x_1^j, \ldots, x_6^j)$, which is the direct sum of two cyclic groups. It follows that the permutation $\delta = (x_0^i, x_1^i, \ldots, x_6^i)(x_0^j, x_1^j, \ldots, x_6^j)$ is 2-orbit-compatible with A, and since $\delta \notin A$, A is not 2-orbit-closed, as claimed. Thus we have proved

Lemma 1.3 *Let A be the permutation group described above. Then A is abelian and satisfies the conditions* (i) *and* (ii) *of Theorem* 1.1*. Yet, A is not 2-orbit-closed, neither 2-closed, nor 2^*-closed, and in particular, it is not the automorphism group of any graph.*

This specific example may be generalized in many ways, showing that both Theorem 1.1 and Lemma 1.2 are false. Our research in the area leads to the following results.

2 Our results

For a permutation group A, whose orbits are X_i, $i = 1, \dots, n$, we define the graph $\mathcal{G}(A)$ whose vertices are orbits of A, and two orbits X_i and X_j are adjacent if and only if the factor groups A_i/A_i^j and A_j/A_j^i are not abstractly isomorphic to Z_2^n for any $n \geq 0$. (One may prove that these factor groups are in fact isomorphic, so the condition may be restricted to one of them).

Theorem 2.1 *Let A be an abelian 2-orbit-closed permutation group. Then A is 2^*-closed if and only if every isolated orbit O in $\mathcal{G}(A)$ has the cardinality 2^n for some $n \geq 0$ and the restriction of A to O is permutation isomorphic to the regular action of Z_2^n.*

In addition we can prove that if A is 2^*-closed, then A is the automorphism group of the complete graph whose edges are colored with not more than 4 colors. For 2-closed groups we have the following characterization.

Theorem 2.2 *An abelian permutation group A is 2-closed if and only if it is 2-orbit-closed. In such a case there exist an edge-colored complete digraph Γ whose edges are colored with not more than 4 colors such that $A = Aut(\Gamma)$.*

The number 4 in the theorem is sharp. We have examples of edge-colored graphs and digraphs with abelian automorphism groups that cannot be interpreted as the automorphism group of a complete (di)graph whose edges are colored with less than 4 colors. We have also results on the abelian permutation groups represented by the automorphism groups of simple graphs and digraphs, but they involve many more technical details. Finally, it is worth to notice that there are non-abelian permutation groups that are 2-orbit-closed but not 2-closed.

References

[1] B. Alspach, *On constructing the graphs with a given permutation group as their group*, Proc. Fifth Southeastern Conf. Combin., Graph Theory and Computing (1974), 187-208.

[2] L. Babai, *Finite digraphs with given regular automorphism groups*, Period. Math. Hungar. **11** n. 4 (1980), 257-270.

[3] F. Dalla Volta, J. Siemons *Orbit equivalence and permutation groups defined by unordered relations*, J. Algebr. Comb. (2012) 35, 547-564.

[4] R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compositio Math., 6 (1938), 239-250.

[5] C. D. Godsil, *GRR's for non-solvable groups*, in: "Algebraic Methods in Graph Theory", Szeged (Hungary) 1978, Colloq. Math. Soc. Janos Bolyai 25, (North–Holland, Amsterdam 1981), 221–239.

[6] M. Grech, *Graphical cyclic permutation groups.* Discrete Math. **337** (2014), 25-33

[7] M. Grech, A. Kisielewicz, *Graphical representations of cyclic permutation groups*, to appear.

[8] W. Imrich, *Graphen mit transitiver Automorphismengruppe*, Monatsh. Math. 73(1969), 341-347.

[9] A. Kisielewicz, *Supergraphs and graphical complexity of permutation groups*, Ars Combinatoria, 101 (2011), 193-207.

[10] D. König, Theorie der Endlichen und Unendlichen Graphen, Akademische Verlags-gesellschaft, Leipzig, 1936.

[11] S. P. Mohanty, M. R. Sridharan, S. K. Shukla, *O*n cyclic permutation groups and graphs, J. Math. Phys. Sci. **12** (1978), no. 5, 409–416,

[12] S. P. Mohanty, M. R. Sridharan, S. K. Shukla, *G*raphical cyclic permutation groups, Combinatorics and graph theory (Calcutta, 1980), 339 – 346, Lecture Notes in Math., 885, Springer, , Berlin - New York, 1981,

[13] J. Morris, *Automorphism Groups of Circulant Graphs - a Survey*, in: Bondy A., Fonlupt J., Fouquet JL., Fournier JC., Ramrez Alfonsn J.L. (eds) Graph Theory in Paris. Trends in Mathematics. Birkhuser Basel 2006, pp. 311-325.

[14] H. Wielandt, *Permutation groups through invariant relation and invariant functions*, in "Mathematische Werke/Mathematical works, vol. 1, Group theory", Bertram Huppert and Hans Schneider (Eds.), Walter de Gruyter Co., Berlin, 1994, pp. 237-266.

[15] A. Z. Zelikovskij, *Königs problem for Abelian permutation groups*, Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk 5 (1989), 34-39.