On spectra of weighted graphs of order ≤ 5

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Abstract

The problem of characterizing the real spectra of weighted graphs is only solved for weighted graphs of order $n \leq 4$. We overview these known results, that come from the context of nonnegative matrices, and give a new method to rule out many unresolved spectra of size 5.

Keywords: Weighted graphs, Real spectra, Adjacency matrix, Symmetric nonnegative inverse eigenvalue problem.

A **weighted digraph** G is a triplet (V, E, w) where V is a nonempty finite set, $E \subset V \times V$ and $w\colon E \rightarrow \mathbb{R}^+$ is a positive real map on E. The elements of V and E are called **vertices** and **arcs** respectively; the values of the map w are called **weights**. The **order** of a digraph is the number of vertices. The **adjacency matrix** of a weighted digraph (V, E, w) with $V = \{v_1, \ldots, v_n\}$ is the matrix $A = (a_{ij})_{i,j=1}^n$ where $a_{ij} = w(v_i, v_j)$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise. The **spectrum** and the **characteristic polynomial** of a weighted digraph are those of its adjacency matrix. In a similar way, a **weighted** (undirected) **graph** can be defined. In this case, the adjacency matrix is symmetric.

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A nonnegative matrix can be seen as the adjacency matrix of a weighted digraph and a symmetric nonnegative matrix as the adjacency matrix of a weighted graph. In this way, nonnegative matrices and weighted digraphs, as well as symmetric nonnegative matrices and weighted graphs, can be considered as equivalent objects.

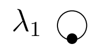
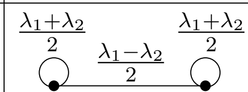
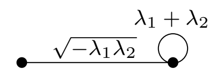
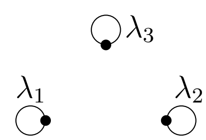
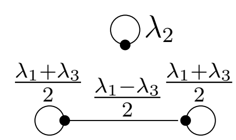
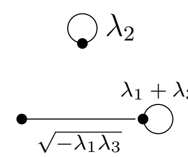
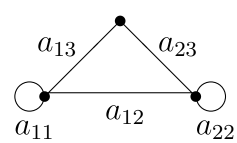
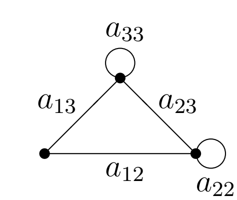
For a family of complex numbers $\sigma = \{\lambda_1, \dots, \lambda_n\}$, repeats allowed, to be the spectrum of a weighted digraph with adjacency matrix A, a number of necessary conditions are known. The most basic of these follow from the fact that a nonnegative matrix has real entries and nonnegative trace, and from the Perron-Frobenius theory of nonnegative matrices:

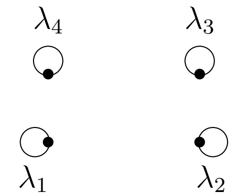
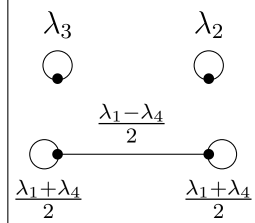
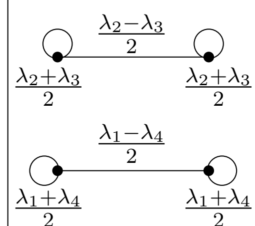
- σ is closed under complex conjugation;
- the trace of A is nonnegative (the trace condition): $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i \geq 0$;
- the moments of σ of all orders are nonnegative, where the **moment of order k** of σ is the number $s_k(\sigma) = \sum_{i=1}^{n} \lambda_i^k = \mathrm{Tr}(A^k), \quad k \geq 1$;
  (Note that the condition $s_1 = 0$, *i.e.* trace 0, means that the weighted graphs considered have no loops.)
- the spectral radius ρ of A is in σ (the Perron condition); without lose of generality, this one may be taken to be λ_1: $\lambda_1 \geq |\lambda_i|, \quad i = 2, \dots n$.

Johnson and, independently, Loewy and London obtained the first non trivial necessary conditions: $(s_j(\sigma))^m \leq n^{m-1} s_{jm}(\sigma), \quad j,\, m = 1, 2, \dots$.

The **RNIEP** (Real Nonnegative Inverse Eigenvalue Problem) is the problem of characterizing all possible real spectra of weighted digraphs. If in the RNIEP we require the digraph to be a graph, we have the **SNIEP** (Symmetric Nonnegative Inverse Eigenvalue Problem). These problems come from the context of nonnegative matrices.

For a long time it was thought that both problems were equivalent, but in 1996 Johnson-Laffey-Loewy [2] set out that both problems are different and in 2004 Egleston-Lenker-Narayan [1] proved that they are different for spectra of size greater than or equal to 5. A complete solution of both problems is known only for spectra of size $n \leq 4$. For these n's the most basic necessary conditions are also sufficient. That is, the trace and the Perron conditions characterize both problems. The next table shows the adyacency matrices and the weighted graphs associated to $\sigma = \{\lambda_1 \geq \cdots \geq \lambda_n\}$ for $n \leq 4$. Most of these constructions are due to Fiedler.

n	Adjacency matrix	Weighted graph
1	(λ_1)	
2	$\begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1-\lambda_2}{2} \\ \frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} \end{pmatrix}$	
	$\lambda_2 \geq 0, \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	
	$\lambda_2 < 0, \quad \begin{pmatrix} 0 & \sqrt{-\lambda_1\lambda_2} \\ \sqrt{-\lambda_1\lambda_2} & \lambda_1+\lambda_2 \end{pmatrix}$	
3	$\lambda_3 \geq 0, \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$	
	$\lambda_2 \geq 0 > \lambda_3, \begin{pmatrix} \frac{\lambda_1+\lambda_3}{2} & \frac{\lambda_1-\lambda_3}{2} & 0 \\ \frac{\lambda_1-\lambda_3}{2} & \frac{\lambda_1+\lambda_3}{2} & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$	
	$\lambda_2 \geq 0 > \lambda_3, \begin{pmatrix} 0 & \sqrt{-\lambda_1\lambda_3} & 0 \\ \sqrt{-\lambda_1\lambda_3} & \lambda_1+\lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$	
	$\lambda_1 \geq 0 > \lambda_2, \begin{pmatrix} \frac{\lambda_1+\lambda_2+\lambda_3}{2} & \frac{\lambda_1+\lambda_2-\lambda_3}{2} & \sqrt{\frac{-\lambda_1\lambda_2}{2}} \\ \frac{\lambda_1+\lambda_2-\lambda_3}{2} & \frac{\lambda_1+\lambda_2+\lambda_3}{2} & \sqrt{\frac{-\lambda_1\lambda_2}{2}} \\ \sqrt{\frac{-\lambda_1\lambda_2}{2}} & \sqrt{\frac{-\lambda_1\lambda_2}{2}} & 0 \end{pmatrix}$	
	$\lambda_1 \geq 0 > \lambda_2$ $\begin{pmatrix} 0 & \sqrt{-(\lambda_1+\lambda_2)\lambda_3} & \sqrt{\frac{-\lambda_1\lambda_2\lambda_3}{\lambda_1+\lambda_2-\lambda_3}} \\ \sqrt{-(\lambda_1+\lambda_2)\lambda_3} & \lambda_1+\lambda_2+\lambda_3 & \sqrt{\frac{-(\lambda_1+\lambda_2)\lambda_1\lambda_2}{\lambda_1+\lambda_2-\lambda_3}} \\ \sqrt{\frac{-\lambda_1\lambda_2\lambda_3}{\lambda_1+\lambda_2-\lambda_3}} & \sqrt{\frac{-(\lambda_1+\lambda_2)\lambda_1\lambda_2}{\lambda_1+\lambda_2-\lambda_3}} & \lambda_2 \end{pmatrix}$	

n	Adjacency matrix	Weighted graph
4	$\lambda_4 \geq 0, \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$	
	$\lambda_3 \geq 0 > \lambda_4, \begin{pmatrix} \frac{\lambda_1+\lambda_4}{2} & \frac{\lambda_1-\lambda_4}{2} & 0 & 0 \\ \frac{\lambda_1-\lambda_4}{2} & \frac{\lambda_1+\lambda_4}{2} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$	
	$\lambda_2 \geq 0 > \lambda_3$ & $\lambda_2+\lambda_3 \geq 0$ $\frac{1}{2}\begin{pmatrix} \lambda_1+\lambda_4 & \lambda_1-\lambda_4 & 0 & 0 \\ \lambda_1-\lambda_4 & \lambda_1+\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_2+\lambda_3 & \lambda_2-\lambda_3 \\ 0 & 0 & \lambda_2-\lambda_3 & \lambda_2+\lambda_3 \end{pmatrix}$	
	$\lambda_2 \geq 0 > \lambda_3$ & $\lambda_2+\lambda_3 < 0$ or $\lambda_1 \geq 0 > \lambda_2$ UDU^t with $U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, D = \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$	K_4

Spectra of size 5 for weighted graphs are not characterized and this problem has proven a very challenging one. Note that when we consider digraphs and graphs (both unweighted) their spectra are completely characterized if one is able to calculate all the spectra of the matrices with 0's and 1's of a fixed size.

In what follows we focus our attention in the SNIEP for $n = 5$. For this n there are two cases where the SNIEP is characterized:

Theorem 0.1 (Spector [8], 2011) *Let* $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ *and* $s_k(\sigma) = \sum_{i=1}^{5} \lambda_i^k$. *Suppose* $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq -\lambda_1$ *and* $s_1(\sigma) = 0$. *Then* σ *is the spectrum of a weighted graph if and only if the following conditions hold:*

(i) $\lambda_2 + \lambda_5 \leq 0$,

(ii) $s_3(\sigma) \geq 0$.

Theorem 0.2 (Loewy-Spector [6], 2017) *Let* $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ *with* $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$ *and* $\sum_{i=1}^{5} \lambda_i \geq \frac{1}{2}\lambda_1$. *Then,* σ *is the spectrum of a weighted graph if and only if the following conditions hold:*

(i) $\lambda_1 = \max_{\lambda\in\sigma} |\lambda|$,

(ii) $\lambda_2 + \lambda_5 \leq \sum_{i=1}^{5} \lambda_i$,

(iii) $\lambda_3 \leq \sum_{i=1}^{5} \lambda_i$.

McDonald-Neumann [7], Egleston-Lenker-Narayan [1] and Loewy-McDonald [5] in their works give a detail discussion of many parts of the case $n = 5$ for positive trace, that is, for weighted graphs with loops.

It is common to study spectra of size 5 considering the number of positive eigenvalues. When there are 1, 4 or 5 positive eigenvalues the answer for the SNIEP is straightforward. When there are just 2 positive eigenvalues, it has been proved by Loewy, for a general n, that "partitioned majorization" is sufficient for the SNIEP, *i.e.*, if the nonpositive eigenvalues $\lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_n$ may be partitioned into 2 subfamilies such that the larger sum of the absolute values in one subfamily is no more than λ_1, and $\lambda_1+\lambda_2$ is at least $|\lambda_3|+\cdots+|\lambda_n|$. For $n \leq 5$, this condition is also necessary. Note that for $n > 5$ this is not true: the family $\{7, 5, -1/2, -7/2, -4, -4\}$, which is the spectrum of a weighted graph [3, Example 8 with $\delta = 1/2$], is not partitionable.

When $n = 5$, this leaves the unresolved case:

$$\lambda_1 > \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5, \qquad \lambda_1 + \lambda_5 \geq 0,$$
$$\sum_{i=1}^{5} \lambda_i > 0 \qquad \text{and} \qquad \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0.$$

The last inequality may be assumed, as, otherwise, $\{\lambda_1, \lambda_2, \lambda_4, \lambda_5\}$ and $\{\lambda_3\}$ would be the spectra of a weighted graph. Thus far, none of these cases has been resolved, except those for which translation by $-\left(\frac{1}{5}\sum \lambda_i\right) I$, leads to a spectrum of a weighted graph without loops.

We study spectra with single spectral radius, the other two positive eigenvalues equal and the two negative eigenvalues also equal. After normalization, the spectra studied are of the form $\{1, a, a, -(a+d), -(a+d)\}$. We give a new method [4], based upon the eigenvalue interlacing inequalities for symmetric matrices, to rule out many unresolved spectra with 3 positive eigenvalues:

Theorem 0.3 *Let* $0 < a, d$ *satisfy* $a + d, 2d < 1 < a + 2d$. *If* $2(a + d)^3 \geq$

$1 + a^3 + (a + 2d - 1)^3$, *then* $\{1, a, a, -(a+d), -(a+d)\}$ *is not the spectrum of a weighted graph of order 5.*

This new method shows that the family $\sigma = \{6, 3, 3, -5, -5\}$, which is the spectrum of a weighted digraph, is not the spectrum of a weighted graph. This spectrum, after normalization, corresponds to $a = \frac{1}{2}$ and $d = \frac{1}{3}$, in which case

$$2\left(\frac{5}{6}\right)^3 \geq 1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{6}\right)^3$$

and the condition of the theorem is satisfied. So σ is not the spectrum of a weighted graph. The "possible symmetric realization" of this spectrum has been intensively studied by many authors in the field of nonnegative matrices during the last 20 years.

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