The Fortress Problem in Terms of the Number of Reflex and Convex Vertices. A 3D objects scanning application

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Abstract

This work focuses on the visibility of exterior of a polygon. We show a lower bound for the Fortress Problem applied to polygons P having n vertices in terms of the number of reflex vertices, the number of convex vertices and the number of pockets that are found when the convex hull is made on P. The results are related to the task of the geometric data acquisition for architectural surveys through techniques such as laser scanner.

Keywords: art gallery theorem, fortress problem, laser scanner

1 Introduction

Art Gallery Problems [1] have become a significant area of study in computational geometry, as they are not only of theoretical interest, but also play a central role in visibility problems [2] [3].

However, it has been not found any research about where to locate guards in order to watch the border of a polygon from the exterior side. This variant has a straight application in the Architectural Graphic Expression. Studying the placement of exterior guards helps to manage point clouds for building scanners in an efficient way.

Among the different versions of the Art Gallery Problem the most linked one with the aforementioned problem is the Fortress Problem, whose purpose is to guard the exterior region of a simple polygon from its border. Derick Wood posed this problem and it was solved by O'Rourke and Wood in 1983 [4], who showed that $\lceil \frac{n}{2} \rceil$ vertex guards are sometimes necessary and always sufficient to cover an n-vertex polygon. A tight bound of $\lceil \frac{n}{3} \rceil$ point guards was given by O'Rourke and Aggarwal [4].

In this work we give a lower bound for the Fortress Problem, providing the function $G(k, r, c)$ for the vertex guard number in terms of the number $k \geq 0$ of pockets and the size of each one, $r \geq 0$, of reflex vertices and the number, $c \geq 3$, of convex vertices of $P(n = r + c)$ following the work achieved by Justin Iwerks and Joseph S.B. Mitchel [2].

The results are related to the concept of *cooperative-guards*, that was applied to the Fortress Problem by Pawel Zylinski [5].

This paper assumes that the vertices of polygons are in general position, i.e. no three vertices are collinear. Let P be a polygon defined by a sequence of vertices, $v_0, v_1, ..., v_{n-1}$ in a clockwise order. Each connected region inside its convex hull but exterior to the polygon is called a *pocket* [6]. All the vertices of P may not be vertices of its convex hull $CH(P)$. The vertices of P, which are not vertices of $CH(P)$ are termed as *notches*. The convex hull edges

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that connect two nonadjacent vertices of P are called *Bridges*. The vertices corresponding to a bridge are the *entry points* of that pocket [7].

The vertices can be distinguished between two different categories: *reflex* vertices r (those whose interior angle is greater than π) and *convex* vertices c (those whose interior angle is at most π). Each k_i pocket contains $r_{i_1}, ..., r_{i_l}$ for $l \geq 1$ vertices, and $c_{i_1}, ..., c_{i_h}$ for $h \geq 2$ of which two are entry points. The rest of vertices of the polygon not contained in any pocket are convex vertices, denoted by $\bar{c_j}$ where $\bar{c} = |V(CH(P))| - 2k$, notice that those vertices belong to the convex hull. Consider the following definition: given a polygon P, $g(P)$ is the minimum number of guards that fully cover the exterior of a polygon P and $G(k, r, c) = max\{G(k, r, c) :$ P is a polygon with k pockets, r reflex vertices and c convex vertices$\}$.

2 Lower Bound Construction

A theorem of Iwerks and Mitchell (Theorem 2 of [2]) shows a tight combinatorial bound in terms of the number of reflex vertices and the number of convex vertices for the Art Gallery Problem (internal surveillance). Here we observe as this result is generalized to the Fortress Problem, in terms of the number of pockets and its size, reflex vertices and convex vertices.

Theorem 2.1 *Given a polygon P with n vertices, having k pockets, r reflex vertices and c convex vertices ($n = r + c$). Then,*

$$G(k, r, c) \leq \lceil \frac{\bar{c} + 2k}{2} \rceil + \sum_{i=1}^{k} G_i(c_i, 2 + r_i)$$

where,

$$(1) \qquad G_i(c_i, 2 + r_i) = \begin{cases} 0 & \text{if } \mathrm{c_i} = 0 \\ c_i & \text{if } \mathrm{c_i} \leq \lfloor \frac{\mathrm{r_i}+2}{2} \rfloor \\ \lfloor \frac{c_i+r_i+2}{3} \rfloor & \text{if } \lfloor \frac{\mathrm{r_i}+2}{2} \rfloor < \mathrm{c_i} < 5(\mathrm{r_i}+2) - 12 \\ 2(r_i + 2) - 4 & \text{if } \mathrm{c_i} \geq 5(\mathrm{r_i}+2) - 12 \end{cases}$$

$\bar{c} = n_{ch} - 2k$

$n_{ch} = |V(CH(P))|$

$c_i =$ *convex vertices in pocket i*

r_i = *reflex vertices in pocket* i

The proof relies on a two-step construction lower bound: (i) number guards that fully cover the exterior of $CH(P)$; (ii) number of guards to cover the pockets according to the number of its convex and reflex vertices.

(i) Guarding the exterior region of the convex hull.

Given the convex hull $CH(P)$ with n_{ch} vertices, they can be distinguished between the entry points and the rest of the convex vertices $\bar{c}$, then $n_{ch} = 2k + \bar{c}$ (see Fig. 1).

A convex polygon requires $\lceil \frac{n}{2} \rceil$ to cover the exterior side [4]. Replacing the value of n by n_{ch} it is got that the exterior of $CH(P)$ can be guarded with $\lceil \frac{\bar{c}+2k}{2} \rceil$ guards in vertices.

(ii) Guarding the pockets.

Guarding the interior region of a pockets it is considered as guarding the interior of a polygon k_i with n_i vertices. Whefore it is distinguished several possibilities, depending of the pocket composition. The different cases are shown as follows:

- Case 1: $c_i = 0$.

  In this case, the pocket is only composed by the two entry points and $r_i > 0$ vertices, being the pocket a convex polygon. Insomuch as a guard has been previously placed in one of the two entry points, it is not necessary to add an extra guard: $G_i(c_i, 2 + r_i) = 0$.

  See example: pocket k_1 in Fig. 1.
- Case 2: $c_i \leq \lfloor \frac{r_i+2}{2} \rfloor$.

  O'Rourke [4] showed that $G(r, c) = r$, if $1 \leq r \leq \lfloor \frac{c}{2} \rfloor$ for guarding the interior of an art gallery.

  Notice in a pocket k_i the reflex vertices are called c_i, and the convex vertices are the 2 entry points and the r_i vertices, then $G_i(c_i, 2+r_i) = c_i$.

  See example: pocket k_2 in Fig. 1.
- Case 3: $\lfloor \frac{r_i+2}{2} \rfloor < c_i < 5(r_i + 2) - 12$.

  Iwerks et all proved in [2] that given a simple polygon P with r vertices and c convex vertices $(n = r + c)$, $\lfloor \frac{n}{3} \rfloor$ point guards are sometimes necessary and always sufficient to cover P when $\lfloor \frac{c}{2} \rfloor < r < 5c - 12$. When this polygon is a pocket, the convex vertices are $r_i + 2$ and the reflex vertices are c_i, given as result that: $G_i(c_i, 2 + r_i) = \lfloor \frac{c_i+r_i+2}{3} \rfloor$

  See example: pocket k_3 in Fig. 1.
- Case 4: $c_i \geq 5(r_i + 2) - 12$.

  Addario-Berry et al. [8] proved that $2c - 4$ points guards are sometimes necessary and always sufficient to cover a simple polygon P with c convex vertices, when $r \geq 5c - 12$.

Replacing the value of c by $r_i + 2$ and the value of r by c_i, we get: $G_i(c_i, 2 + r_i) = 2(r_i + 2) - 4$.

See example: pocket k_4 in Fig. 1.

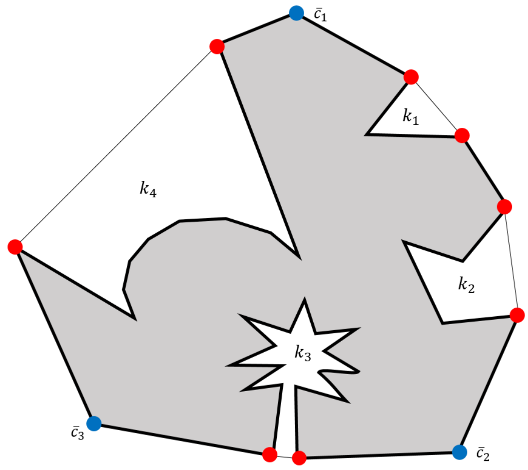

Fig. 1: Convex hull $CH(P)$ of a polygon P, with $k = 4$ pockets, $2k = 8$ entry points, and three $\bar{c}$ convex vertices.

3 Application and further research.

For architectural surveyment tasks it is really important to plannify the points from which geometric data acquisition will be executed. The results are used to design an algorithm able to define the exterior areas to the polygon from which the entire border of the polygon is guarded. The input of this algorithm is the set of vertice coordinate of a polygon. The output is a sufficient and sometimes necessary number of guards to watch the border of the polygon from the exterior side, and the area along which a guard can patrol to guarantee the whole surveillance.

In future works, it will be taken into account the concept of *cooperative guards*, proposed by Liaw et al [9], to design a connected net of points of view. For a guard set S it is defined the visibility graph $VG(S)$ as the graph whose set of vertices is S, and $v_1, v_2 \in S$ define an edge if they see each other. The guard set S is said to be cooperative if the graph $VG(S)$ is connected.

In addition, in further researches it will be desirable to get a uniform estimation of the bound for some general parameters of the polygon, without loosing the application to the topographic tasks. Furthermore, it can be possible to find a lower bound construction analyzing different typologies of polygons.

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