On graphs with the same truncated U-polynomial and the U-polynomial for rooted graphs

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Abstract

In this abstract, we construct explicitly, for every k, pairs of non-isomorphic trees with the same restricted U-polynomial; by this we mean that the polynomials agree on terms with degree at most k. The construction is done purely in algebraic terms, after introducing and studying a generalization of the U-polynomial to rooted graphs.

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1 Introduction

The chromatic symmetric function (c.s.f.) [Sta95] and the U-polynomial [NW99] are powerful graph invariants as they generalize other interesting invariants like, for instance, the chromatic polynomial, the matching polynomial and the Tutte polynomial. It is well known that the c.s.f. and the U-polynomial are equivalent when restricted to trees, and there are examples of graphs with cycles having the same U-polynomial (resp. the same c.s.f.). However, it is an open question to know whether there exist non-isomorphic trees with the same c.s.f (or, equivalently, the same U-polynomial). The negative answer to the latter question, that is, the assertion that two trees that have the same c.s.f must be isomorphic, is sometimes referred to in the literature as *Stanley's (tree isomorphism) conjecture*. This conjecture has been so far verified for trees up to 29 vertices [HJ18] and also for some classes of trees, most notably caterpillars [APZ14,LS14] and spiders [MMW08]. A natural simplification for Stanley's conjecture is to truncate the U-polynomial, construct non-isomorphic trees with the same truncated U-polynomial and study these examples to better understand the picture for the non-truncated U-polynomial. In a previous work [APdMZ17], the authors studied the truncation of U-polynomial by simply restricting to terms with degree lower than a certain fixed k, and showed the existence of the examples in this setting. This result was based on a remarkable connection between such examples and solutions to the Prouhet-Tarry-Escott problem in number theory.

The drawback of the above construction is that it is very difficult to give explicit solutions to the Prouhet-Tarry-Escott problem. In this paper, we give an explicit and simple construction of examples of non-isomorphic trees with the same truncated U-polynomial for every k. These examples coincide with the examples already found by Smith, Smith and Tian [SST15] for $k = 2, 3, 4, 5$, which leads us to conjecture that for every k our construction yields the smallest non-isomorphic trees with the same truncated U-polynomial.

The main tool for proving our result is the introduction and study of the generalization of the U-polynomial to rooted graphs. It turns out that the rooted U-polynomial distinguishes rooted trees, and it is equivalent to the rooted polychromate introduced by Bollobás and Riordan [BR00]. It is also very close to a recently introduced rooted version of the c.s.f. [Paw18]. The key fact for us is that the rooted U-polynomial exhibits nice formulas when applied to products of rooted graphs. These formulas together with some non-commutativity is what allows our constructions to work.

2 The rooted U-polynomial

First, we recall the definition of the U-polynomial introduced by Noble and Welsh [NW99]. We consider graphs where we allow loops and multiple edges.

Let $G = (V, E)$ be a graph. Given $A \subseteq E$, the restriction $G|_A$ of G to A is the subgraph of G obtained from G after deleting every edge that is not contained in A (but keeping all the vertices). The *rank* of A is defined as $r(A) = |V| - k(G|_A)$, where $k(G|_A)$ is the number of connected components of $G|_A$. The *partition induced by* A, denoted by $\lambda(A)$, is the partition of $|V|$ whose components are the sizes of the connected components of $G|_A$.

Let y be an indeterminate and $\mathbf{x} = x_1, x_2, \ldots$ be an infinite set of conmuting indeterminates that commute with y. Given any partition λ, define $\mathbf{x}_\lambda := x_{\lambda_1} \cdots x_{\lambda_l}$. The *$U$-polynomial* of a graph G is defined as

$$U_G(\mathbf{x}, y) = \sum_{A \subseteq E} \mathbf{x}_{\lambda(A)} (y-1)^{|A|-r(A)}. \tag{1}$$

Given an integer k, the U_k-polynomial of G is defined the same way as in (1) but with the summation ranging over all $A \subseteq E$ with less than k elements. We note that the U-polynomial is the specialization of the W-polynomial which is defined for weighted graphs. The key fact about the W-polynomial is that it satisfies a *deletion-contraction formula*, for details see [NW99]. A *rooted graph* is a pair (G, v_0), where G is a graph and v_0 is a vertex of G that we call the *root* of G. Given $A \subseteq E$, define $\lambda_r(A)$ to be the size of the component of $G|_A$ that contains the root v_0, and $\lambda_-(A)$ to be the partition induced by the sizes of all the other components. The *rooted U-polynomial* is

$$U^r_{(G,v_0)}(\mathbf{x}, y, z) = \sum_{A \subseteq E} \mathbf{x}_{\lambda_-(A)} z^{\lambda_r(A)} (y-1)^{|A|-r(A)}. \tag{2}$$

We often write G instead of (G, v_0) when v is clear, so we will write $U^r(G)$ and $U(G)$ for convenience. If we compare $U^r(G)$ with $U(G)$, then we see that for each term of the form $\mathbf{x}_\lambda y^n z^m$ appearing in $U^r(G)$ there is a corresponding term of the form $\mathbf{x}_\lambda y^n x_m$ in $U(G)$. If we use the notation $(P)^*$, where P is a polynomial in z, to denote the result of substituting z^n by x_n for all $n \in \mathbb{N}$, we can conclude that for every rooted graph (G, v) we have

$$(U^r(G))^* = U(G). \tag{3}$$

We now ask whether the U^r-polynomial distinguishes rooted trees up to isomorphism. To see this, we first recall that the polychromate is an invariant introduced in [Bry81] and later found to be equivalent to the U-polynomial in [Sar00]. In [BR00] the authors proved that a rooted version of the polychromate distinguishes rooted trees up to isomorphism. We show:

Theorem 2.1 *The U^r-polynomial is equivalent to the rooted polychromate. In particular, it distinguishes rooted trees up to isomorphism.*

Let (G, v) and (H, v') be two rooted graphs. Define $G \odot H$ as the rooted graph obtained from the disjoint union of G and H after identifying v and v'. Clearly $G \odot H = H \odot G$. Define $G \cdot H$ as the rooted graph obtained from the disjoint union of G and H then adding the edge vv'. The root of $G \cdot H$ is v. In this case, $G \cdot H \neq H \cdot G$ as rooted graphs.

Lemma 2.2 *Let G and H be two rooted graphs. We have*

$$U^r(G \odot H) = \frac{1}{z} U^r(G) U^r(H), \quad \text{and} \tag{4}$$

$$U^r(G \cdot H) = U^r(G)(U^r(H) + U(H)). \tag{5}$$

3 Non-isomorphic trees with the same truncated U-polynomial

We start by defining two sequences of rooted trees. Let us denote the path on three vertices, rooted at the central vertex, by A_0 and the path on three vertices, rooted at one of the leaves, by B_0. The trees A_k and B_k for $k \in \mathbb{N}$ are defined inductively as follows:

$$A_k := A_{k-1} \cdot B_{k-1} \quad \text{and} \quad B_k := B_{k-1} \cdot A_{k-1}.$$

We first observe that A_0 and B_0 are isomorphic as unrooted trees but not isomorphic as rooted trees, which means that they have different U^r. In fact, a direct calculation shows that $\Delta_0 := U^r(A_0) - U^r(B_0) = x_1 z^2 - x_2 z$. By applying Lemma 2.2 we deduce:

Proposition 3.1 *For all $k \in \mathbb{N}$, the trees A_k and B_k are isomorphic but not rooted-isomorphic. Moreover, $U^r(A_k) - U^r(B_k) = \Delta_0 P_k$, where $P_k := U(A_0)U(A_1)\cdots U(A_{k-1})$.*

Observe that all the terms of P_k have degree at least k. Now we can state our main result.

Theorem 3.2 *Given $k, l \in \mathbb{N}$, let*

$$Y_{k,l} = (A_k \odot A_l) \cdot (B_k \odot B_l) \quad \text{and} \quad Z_{k,l} = (A_l \odot B_k) \cdot (B_l \odot A_k). \tag{6}$$

Then, $Y_{k,l}$ and $Z_{k,l}$ are not isomorphic and $U_{k+l+2}(Y_{k,l}) = U_{k+l+2}(Z_{k,l})$.

Proof. The proof is based in the repeated application of the following lemma, which is a corollary of Lemma 2.2 and Proposition 3.1.

Lemma 3.3 *Let T be a rooted tree and i an integer. Then*

$$U(A_i \odot T) - U(B_i \odot T) = P_i \mathcal{D}(T), \tag{7}$$

where

$$\mathcal{D}(T) = x_1(zU_r(T))^* - x_2 U(T). \tag{8}$$

In particular all the terms in $\mathcal{D}(T)$ have degree at least 2.

We start by applying the deletion-contraction formula to the edge corresponding to the concatenation operation in $Y_{k,l}$ and $Z_{k,l}$; it is easy to see that

$$U(Y_{k,l}) - U(Z_{k,l}) = U(A_k \odot A_l)U(B_k \odot B_l) - U(A_l \odot B_k)U(B_l \odot A_k), \tag{9}$$

since after contracting the respective edges we get isomorphic weighted trees.

By applying Lemma 3.3 three times[5] and then plugging the results into (9), after some appropriate algebraic manipulations, we get

$$U(Y_{k,l}) - U(Z_{k,l}) = P_l\Big(\big(\mathcal{D}(A_k) - \mathcal{D}(B_k)\big)U(B_l \odot A_k) + \mathcal{D}(A_k)\mathcal{D}(B_k)P_k\Big) \tag{10}$$

Using (8) we compute $\mathcal{D}(A_k) - \mathcal{D}(B_k) = x_1 P_k (z\Delta_0)^* = x_1(x_1x_3 - x_2^2)P_k$, and substituting this into (10) yields

$$U(Y_{k,l}) - U(Z_{k,l}) = P_l P_k\Big((x_1^2x_3 - x_1x_2^2)U(B_l \odot A_k) + \mathcal{D}(A_k)\mathcal{D}(B_k)\Big) \tag{11}$$

This implies that all the terms that appear in the difference have degree at least $l + k + 4$. The conclusion now follows. $\square$

In the cases $(k, l) \in \{(0, 0), (1, 0), (1, 1)\}$ we reobtain the examples of Smith Smith and Tian of non-U_m-unique trees with $m \in \{2, 3, 4\}$[SST15]. In these cases, they computationally showed that the examples were minimal. Hence we propose the following conjecture:

Conjecture 3.4 *For each (k, l), the trees $Y_{k,l}$, $Z_{k,l}$ are a smallest pair of graphs with the property that $U_{k+l+2}(Y_{k,l}) = U_{k+l+2}(Z_{k,l})$.*

[5] With $T = A_k$ and $i = l$ first, then $T = B_k$ and $i = l$ and finally with $T = B_l$ and $i = k$

Since the size of the trees $Y_{k,l}$ and $Z_{k,l}$ increases exponentially in $k+l$, then Conjecture 3.4 implies Stanley's tree-isomorphism conjecture.

References

[APdMZ17] José Aliste-Prieto, Anna de Mier, and José Zamora. On trees with the same restricted U-polynomial and the Prouhet-Tarry-Escott problem. *Discrete Math.*, 340(6):1435–1441, 2017.

[APZ14] José Aliste-Prieto and José Zamora. Proper caterpillars are distinguished by their chromatic symmetric function. *Discrete Mathematics*, 315:158–164, 2014.

[BR00] Béla Bollobás and Oliver Riordan. Polychromatic polynomials. *Discrete Mathematics*, 219(1):1–7, 2000.

[Bry81] Tom Brylawski. Intersection theory for graphs. *J. Combin. Theory Ser. B*, 30(2):233–246, 1981.

[HJ18] S. Heil and C. Ji. On an algorithm for comparing the symmetric chromatic functions of trees. *arXiv preprint arXiv:1801.07363*, 2018.

[LS14] Martin Loebl and Jean-Sébastien Sereni. Isomorphism of weighted trees and stanley's conjecture for caterpillars. *arXiv preprint arXiv:1405.4132*, 2014.

[MMW08] Jeremy L Martin, Matthew Morin, and Jennifer D Wagner. On distinguishing trees by their chromatic symmetric functions. *Journal of Combinatorial Theory, Series A*, 115(2):237–253, 2008.

[NW99] S. D. Noble and D. J. A. Welsh. A weighted graph polynomial from chromatic invariants of knots. *Ann. Inst. Fourier (Grenoble)*, 49(3):1057–1087, 1999. Symposium à la Mémoire de François Jaeger (Grenoble, 1998).

[Paw18] Brendan Pawlowski. Chromatic symmetric functions via the group algebra of s_n. *arXiv preprint arXiv:1802.05470*, 2018.

[Sar00] Irasema Sarmiento. The polychromate and a chord diagram polynomial. *Ann. Comb.*, 4(2):227–236, 2000.

[SST15] Isaac Smith, Zane Smith, and Peter Tian. Symmetric chromatic polynomial of trees. *arXiv preprint arXiv:1505.01889*, 2015.

[Sta95] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.