Satisfaction and Power in Unanimous Majority Influence Decision Models

Xavier Molinero [1]

Department of Mathematics, Technical University of Catalonia. Terrasa, Spain

Fabián Riquelme [2]

Escuela de Ingeniería Civil Informática, Universidad de Valparaíso, Chile

Maria Serna [3]

Computer Science Department and Barcelona Graduate School on Mathematics BGSMath, Technical University of Catalonia. Barcelona, Spain

Abstract

We consider decision models associated with cooperative influence games, the *oblivious* and the *non-oblivious* influence models. In those models the *satisfaction* and the *power* measures were introduced and studied. We analyze the computational complexity of those measures when the influence level is set to unanimity and the rule of decision is simple majority. We show that computing the satisfaction and the power measure in those systems are #P-hard.

Keywords: Decision model, Influence game, Satisfaction, Power, Banzhaf value

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[1] Partially funded by Grant MTM2015-66818-P from MINECO. Email: xavier.molinero@upc.edu

[2] Email: fabian.riquelme@uv.cl

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1 Introduction

A *decision model* $\mathcal{M}$ is a tuple (V, D, q) where $V = \{1, \ldots, n\}$, $D : \{0,1\}^n \to \{0,1\}^n$ is a function, and $0 \leq q \leq n+1$. For a participants' *initial decision* vector $x \in \{0,1\}^n$, the *final decision vector* is $y = D(x)$. The associated *collective decision function* $C_{\mathcal{M}} : \{0,1\}^n \to \{0,1\}$ is defined as $C_{\mathcal{M}}(x) = 1$ iff $|\{i \in V \mid y_i = 1\}| \geq q$. Motivated by the theoretical study of the effects that collective decision-making can have on the participants, *satisfaction* (Sat) and *power* (Pow) measures were defined in [8]. The satisfaction of $i \in V$ is the number of initial decisions for which the final collective decision coincides with the initial decision of i. The power of $i \in V$ is the number of initial decision for which the collective decision changes when i changes its initial decision.

An *influence graph* is a tuple (G, f), where $G = (V, E)$ is a directed graph and f is a labeling function assigning to any vertex a non-negative rational value. Let (G, f) be an influence graph and let $X \subseteq V$. The *activation process*, with initial activation X, at time t, $0 \leq t \leq n$, activates a set of vertices $F^t(X)$ defined as follows: $F^0(X) = X$ and $F^t(X) = F^{t-1}(X) \cup \{i \in V \mid |P_G(i) \cap F^{t-1}(X)| \geq f(i)\}$, for $1 \leq t \leq n$, where $P_G(i) = \{j \in V \mid (j,i) \in E\}$ is the set of *predecessors* of i. The *spread of influence* of X in (G, f) is the set $F(X) = F^n(X)$. As usual, for an undirected graph G, $N(u)$ denotes the set of neighbors of u.

Given (G, f, q, N), where (G, f) is an influence graph with positive labeling function, $0 \leq q \leq N$ and $N \subseteq V(G)$, The associated *oblivious influence model* is the decision model $\mathcal{M}^o(G, f, q, N) = (V(G), D, q)$ where, for $x \in \{0,1\}^n$, $y = D(x)$ is defined as $y_i = 1$ iff $i \in F(X(x) \cap N)$. The associated *non-oblivious influence model* is the decision model $\mathcal{M}^n(G, f, q, N) = (V(G), D, q)$ where $y = D(x)$ is defined as follows. For $x \in \{0,1\}^n$, let $p_i^1(x) = |F(X(x) \cap N) \cap P(i)|$ and $p_i^0(x) = |P(i) \setminus F(X(x) \cap N)|$. For $i \in V(G) \setminus N$, $y_i = 1$ iff $i \in F(X(x))$. For $i \in N$, if for one $z \in \{0,1\}$, $p_i^z(x) \geq f(i)$ and $p_i^{\overline{z}}(x) < f(i)$, we set $y_i = z$ otherwise $y_i = x_i$.

Cooperative *influence games* were introduced in [4] and influence decision models in [5] together with an analysis of the complexity of computing the Sat measure. The computational complexity of the Pow measure was analyzed in [6]. In those papers it is shown that the problem of computing the Sat(i) or Pow(i) are known to be #P-hard, for oblivious and non-oblivious general influence models. Those measures can be computed in polynomial time in strong hierarchical and star influence graphs. However it was left open the

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particular cas ein which the final decision rule is majority.

We analyze here the computational complexity of the Sat and the Pow measures in oblivious and non-oblivious influence decision models in the particular case of *unanimous majority influence models*. In a unanimous majority influence model the graph G is undirected. Besides the nodes require unanimity for a change of opinion, i.e., $f(i) = |N(i)|$, and the final decision is taken by the simple majority rule, i.e., $q = \lfloor \frac{|V|}{2} \rfloor + 1$. We show #P-hardness that computing $\text{Sat}(i)$ or $\text{Pow}(i)$ is #P-hard, for oblivious and non-oblivious unanimous majority influence models.

2 Computing Satisfaction

In order to show #P-hardness we provide reductions from the problem of counting the number of vertex covers of a given graph. The problem is known to be #P-complete for bipartite graphs, even for 3-regular planar bipartite graphs [9].

Our reductions make use of the following construction. Let $G = (V, E)$ be a connected graph with n nodes. The influence game $\Gamma(G) = (G', f, q, N)$ is defined by $G' = (V'E')$ which is a graph on $2n+1$ vertices; $V' = V \cup W \cup \{a, b\}$ where $W = \{w_1, \dots, w_{n-1}\}$ and $E' = E \cup \{(a, w_i) \mid 1 \leq i \leq n-1\}$. $f(u) = |N_G(u)|$, for $u \in V$, $f(u) = 1$, for $u \in W$, $f(a) = n - 1$ and $f(b) = 1$. $q = n + 1$ and $N = V \cup \{a, b\}$. Observe firstly that G' has $2n + 1$ vertices, thus a decision is taken by simple majority rule on the final decision vector. Secondly, as $f(u) = |N_G(u)|$, for $u \in V$, the set of coalitions $X \subseteq V$ having $F(X) = V$ coincides with the set of vertex covers of G.

We can prove the following properties of the winning coalitions in the game $\Gamma(G)$, depending on the decisions of a and b.

Lemma 2.1 *Let $G = (V, E)$ be a connected graph with n and let $\Gamma = \Gamma(G)$. For a coalition S, we have the following characterization. If $a, b \in S$, then S is wining. If $a \notin S$ and $b \in S$, S is winning iff $S \cap V(G)$ is a vertex cover of G. If $a \in S$ and $b \notin S$, S is winning iff $|S \cap V| \geq 1$. If $a, b \notin S$, then S is losing.*

Note that, in the oblivious model, the winning coalitions coincide with the decision vectors in which the collective decision is 1.

Theorem 2.2 *Computing the satisfaction measure is #P-hard for oblivious unanimous majority influence models even on 3-regular bipartite planar graphs.*

Proof. [Sketch] Let G be a graph and consider the influence game $\Gamma(G)$ as described before. Recall that in the oblivious model, the wining coalitions of $\Gamma(G)$ lead to a final decision vector with a majority of 1's and therefore the collective decision is 1. On the other hand, losing coalitions force a majority of 0's in the final decision vector and the collective decision is then set to 0. Recall that in the oblivious model the initial decision of the non-players is disregarded therefore the result is independent of the initial decision of the nodes in W. Thus, taking into account the characterization given in Lemma 2.1 we can prove the following expression for $\textsc{Sat}(b)$,

$$\textsc{Sat}(b) = 2^{n-1}\left[2^n + \#VC(G) + 1 + 2^n\right],$$

where $\#VC(G)$ is the number of vertex covers of G. As computing $\#VC(G)$ is #P-hard even for 3-regular planar bipartite graphs the claim follows. □

For the non-oblivious model the interactions in $\Gamma(G)$ are more complex. We have to take care of those players that change their initial inclination to a 0. Our next result characterize the final collective decision on $\Gamma(G)$ according to the initial decision of a and b in the non-oblivious model.

Lemma 2.3 *Let $G = (V, E)$ be a connected graph with n and let $\mathcal{M}$ be the non-oblivious model associated to $\Gamma(G)$. For an initial decision vector x, we have the following characterization. If $x_a = x_b = 1$, $C(x) = 1$. If $x_a = 0$ and $x_b = 1$, $C(x) = 1$ iff $X(x) \cap V$ is a vertex cover of G. If $x_a = 1$ and $x_b = 0$, if there exists $u \in V$ with $x_u = 0$ and $N_G(u) \subseteq X(x)$, then $C(x) = 1$; otherwise, $C(x) = 0$ iff $(V \setminus X(x))$ is a vertex cover of G. If $x_a = x_b = 0$, $C(x) = 0$.*

For the non-oblivious models, the relationship with the number of vertex covers is less clear. Observe that the condition there exists $u \in V$, $x_u = 0$ and $N_G(u) \subseteq X(x)$ identifies the so-called *i-essential* sets. According to [2] the problem of computing the number of i-essential sets in polynomial time is open. The vertex covers verifying the negated condition are the so-called *total* vertex covers. To the best of our knowledge the complexity of counting the number of total vertex covers is an open problem.

Our hardness result holds for a family of graphs that we call *1-almost bipartite*, those graphs that become bipartite after removing one vertex.

Theorem 2.4 *Computing the satisfaction measure is #P-hard for non-oblivious unanimous majority influence models even on 1-almost 3-regular bipartite planar graphs.*

Proof. [Sketch] Let G be a graph with $n-1$ vertices and consider the graph $G'' = (V'', E'')$ where $V'' = V \cup \{c\}$ and $E'' = E \cup \{(c, u) \mid u \in V\}$ and the influence game $\Gamma(G'')$ using the construction described before. From Lemma 2.3 we can prove that

$$\textsc{Sat}(b) = \left[2^{2n-1} + 2^{n-1}\#VC(G'') + 2^{n-1}\#TVC(G'') + 2^{2n-1}\right],$$

where $\#TVC(G)$ is the number of total vertex covers of G. We can show that $\#TVC(G'') = \#VC(G) + 1$, and therefore, the claim follows. □

As a consequence of the previous result we have the following.

Corollary 2.5 *The problem of computing the number of total vertex covers in a given graph is #P-complete.*

3 Related Results

A family of subsets $\mathcal{W} \subseteq \mathcal{P}(N)$ is said to be *monotonic* when, for any $X \in \mathcal{W}$ and $Z \in \mathcal{P}(N)$, if $X \subseteq Z$, then $Z \in \mathcal{W}$. A *simple game* Γ is given by a tuple $(N, \mathcal{W})$ where N is a finite set of players and $\mathcal{W}$ is a monotonic family of subsets of N. The subsets in $\mathcal{W}$ are the winning coalitions. The Banzhaf value (Bz) measures the proportion of coalitions in which a player plays a critical role, i.e., if he jumps out from a winning coalition leaves a losing coalition [1]. The Rae index (Rae) measures the number of winning coalitions containing player i and the number of losing coalitions in which player i does not participate [7]. In [3] it was shown that, for each player i, $\textsc{Rae}(i) = \textsc{Bz}(i) + 2^{n-1}$. Thus, the computational complexity of the two indices is the same. Computing the $\textsc{Bz}(i)$ is polynomial time solvable for simple games represented by the set of winning coalitions, but it is #P-complete for simple games represented by the set of minimal winning coalitions. The problem is also known to be #P-hard for weighted voting games and influence games.

To any decision model $\mathcal{M} = (V, D, q)$ we can associate the set system $\mathcal{S}_{\mathcal{M}} = \{S \subseteq V \mid C_{\mathcal{M}}(x(X)) = 1\}$. In [5] we shown that when $\mathcal{M}$ is and influence decision models (oblivious or non-oblivious) the family $\mathcal{S}_{\mathcal{M}}$ is monotonic. Therefore, $\Gamma(\mathcal{M}) = (V, \mathcal{S}_{\mathcal{M}})$ is a simple game. Furthermore, the satisfaction measure on $\mathcal{M}$ coincides with the Rae index in $\Gamma(\mathcal{M})$. We can show a linear relationship and the power measure is twice the *Banzhaf value*.

Theorem 3.1 *Let $\mathcal{M} = \mathcal{M}^o(G, f, q, N)$ or $\mathcal{M} = \mathcal{M}^n(G, f, q, N)$ and let $\Gamma = \Gamma(\mathcal{M})$. Then, for $i \in V$, $\textsc{Pow}_{\mathcal{M}}(i) = 2\textsc{Bz}_{\Gamma}(i)$ and $\textsc{Pow}_{\mathcal{M}}(i) = 2(\textsc{Sat}_{\mathcal{M}}(i) - 2^{n-1})$.*

From the last equality, all the results on the complexity of computing satisfaction apply also to the problem of computing power.

Theorem 3.2 *Computing the power measure is #P-hard for oblivious unanimous majority influence models even on 3-regular bipartite planar graphs and for non-oblivious unanimous majority influence models even on 1-almost 3-regular bipartite planar graphs.*

Also the computation of the Rae index or the Banzhaf value, in the simple games corresponding to such families are #P-hard. Thus, our results extend the subfamilies of simple games for which the complexity of the computation of the Banzhaf value is known. We are working towards getting a better understanding of the combinatorial characterization of such games. In particular they include the simple games defined by the family of vertex covers of a minimum size which are of independent interest.

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