Kirchhoff index of the connections of two networks by an edge

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Abstract

In this work we compute the group inverse of the Laplacian of the connections of two networks by and edge in terms of the Laplacians of the original networks. Thus the effective resistances and Kirchhoff index of the new network can be derived from the Kirchhoff indexes of the original networks.

Keywords: Laplacian matrix, Group inverse, Effective Resistances, Kirchhoff index.

1 Introduction

Among the different parameters that have been proposed to quantify structural properties of a network one of the most useful is the *effective resistance* between two vertices of the network and the *total resistance* or *Kirchhoff index* of the network. This concept appears either in electrical circuits and in the Organic Chemistry, where the effective resistance emerged as a better alternative to other parameters used for discriminating among different molecules with similar shapes and structures.

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However, the computation of the effective resistances is not easy, as they are highly sensitive to small perturbations on the network, such as increasing or decreasing the conductance of one edge, adding or deleting some vertex, see for instance [4]. It is known that the effective resistances are a function of the elements of the group inverse of the of the combinatorial Laplacian of the network. Therefore, the bigger the network is, the more difficult is to compute the group inverse of the combinatorial Laplacian. Thus our aim is to find relations between effective resistances and Kirchhoff indexes of the original network and some parts of them; see for instance [4,7] and references therein. This strategy made necessary to extend the concept of effective resistance, defining the effective resistance between any pair of vertices with respect to a nonnegative value and a weight on the vertex set, see [2].

In this paper we consider composite networks built *connecting* two simpler networks by an edge. Since the Laplacian of these composite networks can be divided into blocks, we compute the group inverse of the Laplacian in terms of the inverse of the blocks for a general case. We point out that there exists many results in the literature computing generalized inverse of matrices partitioned into blocks, see for instance [5]. However, since their aim is to provide a very general formulae, they are usually intricate and very difficult to manage them in specific cases. So, in specific frameworks it is more useful to provide, more or less directly, the following result, that can be found in [8].

Theorem 1.1 *The group inverse of* L' *is given by*

$$(\mathsf{L}')^{\dagger} = \begin{pmatrix} \mathsf{X}_1 & \mathsf{C} \\ \\ \mathsf{C}^{\top} & \mathsf{X}_2 \end{pmatrix},$$

where

$$\mathsf{X}_1 = \frac{1}{n_1^2 n_2^2}\left[(n_1 n_2 \mathsf{I}_n + \mathsf{J}_{\mathsf{n_1 n_2}} \mathsf{H}_2^{-1}\mathsf{B}^{\top})\mathsf{M}(n_1 n_2 \mathsf{I}_{\mathsf{n_1}} + \mathsf{B}\mathsf{H}_2^{-1}\mathsf{J}_{\mathsf{n_2 n_1}}) + \mathsf{J}_{\mathsf{n_1 n_2}}\mathsf{H}_2^{-1}\mathsf{J}_{\mathsf{n_2 n_1}}\right],$$

$$\mathsf{X}_2 = \left(\mathsf{I}_{\mathsf{n_2}} - \frac{1}{n_2^2}\mathsf{J}_{\mathsf{n_2}}\right)\mathsf{H}_2^{-1}(\mathsf{H}_2 + \mathsf{B}^{\top}\mathsf{M}\mathsf{B})\mathsf{H}_2^{-1}\left(\mathsf{I}_{\mathsf{n_2}} - \frac{1}{n_2^2}\mathsf{J}_{\mathsf{n_2}}\right),$$

$$\mathsf{C} = -\left[\frac{1}{n_1 n_2}\mathsf{J}_{\mathsf{n_1 n_2}} + \left(\mathsf{I}_{\mathsf{n_1}} + \frac{1}{n_1 n_2}\mathsf{J}_{\mathsf{n_2 n_1}}\mathsf{H}_2^{-1}\mathsf{B}^{\top}\right)\mathsf{M}\mathsf{B}\right]\mathsf{H}_2^{-1}\left(\mathsf{I}_{\mathsf{n_2}} - \frac{1}{n_2^2}\mathsf{J}_{\mathsf{n_2}}\right),$$

the matrix $\mathsf{M} = (\mathsf{H}_1 - \mathsf{B}\mathsf{H}_2^{-1}\mathsf{B}^{\top})^{\dagger}$ *and* $\mathsf{H}_2 = \mathsf{L}_2 + \mathit{diag}(\sum_{i=1}^{m_1} a_{1j}, \ldots, \sum_{i=1}^{m_1} a_{n_2 j})$.

We observe that the application of this result depends on the computation of the inverse of one matrix, H_2 and matrix $\mathsf{M} = (\mathsf{H}_1 - \mathsf{B}\mathsf{H}_2^{-1}\mathsf{B}^\top)^\dagger$, which depends on the nature of the matrices involved. Therefore, for the sake of simplicity we can study cases where these computations can be avoided.

2 Connecting two networks by a bridge

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two networks, with n_1 and n_2 vertices and respective conductances $c_1 > 0$ and $c_2 > 0$. Our aim is to connect them by a new edge e of conductance $a > 0$, and to compute the inverse Laplacian matrix of the new network, L, in terms of the Laplacian matrices L_1 and L_2. For this purpose we consider the corresponding Schrödinger operators with $\lambda = 0$ and constant weights. Let $G = (V, E)$ be the new network, with $V = V_1 \cup V_2$, $E = E_1 \cup E_2 \cup \{e\}$. The order of the new network is $n = n_1 + n_2$. The associated weight of the new network is $\mathsf{u}^\top = (\sqrt{n})^{-1}\mathsf{j}_n = (\mathsf{v}^\top, \mathsf{w}^\top)$. Thus the corresponding kernels of $\mathcal{L}_q^1$ and $\mathcal{L}_q^2$ are the Laplacian matrices L_1 and L_2, and the kernel of the Laplacian operator of the new network is

$$\mathsf{L} = \begin{pmatrix} \mathsf{L}_1 + \mathsf{D}_1 & \mathsf{B} \\ \mathsf{B}^\intercal & \mathsf{L}_2 + \mathsf{D}_2 \end{pmatrix},$$

where $\mathsf{D}_i = diag(a, 0, \dots, 0) \in \mathcal{M}_{n_i \times n_i}$, $i = 1, 2$, and $\mathsf{B} \in \mathcal{M}_{n_1 \times n_2}$ with $(\mathsf{B})_{11} = -a$, and $(\mathsf{B})_{ij} = 0$ otherwise. Now we compute the matrix

$$\mathsf{M} = \mathsf{S}^\dagger = (\mathsf{L}_1 + \mathsf{D}_1 - \mathsf{B}(\mathsf{L}_2 + \mathsf{D}_2)^{-1}\mathsf{B}^\top)^\dagger.$$

First we compute $\mathsf{H}_2^{-1} = (\mathsf{L}_2 + \mathsf{D}_2)^{-1}$ by applying Theorem 3.5. from [3] with only one perturbation $\sigma = \sqrt{a}\varepsilon_1$. Taking into account that $\mathsf{L}_2^\dagger = \mathsf{G}_2 = (g_{ij}^2)$, then

$$\left(\mathsf{H}_2^{-1}\right)_{ij} = g_{ij}^2 - (g_{1i}^2 + g_{1j}^2) + g_{11}^2 + \frac{1}{a}, \quad \text{for any} \quad i, j = 1, \dots, n_2.$$

Then $\mathsf{B}(\mathsf{L}_2 + \mathsf{D}_2)^{-1}\mathsf{B}^\top = diag(a, 0, \dots, 0) = \mathsf{D}_1$, and thus

$$\mathsf{M} = {\mathsf{L}_1}^\dagger = \mathsf{G}_1.$$

Second, observe that $\mathsf{v}\,\mathsf{w}^\top = (1/n)\mathsf{J}_{\mathsf{n_1 n_2}}$ and $\mathsf{w}\,\mathsf{w}^\top = n^{-1}\mathsf{J}_{\mathsf{n_2 n_2}}$; besides if we detone by $\mathsf{u}_\mathsf{k} = (-1, 0, \dots, 0)^\top \in \mathcal{M}_{k \times 1}$, and by $\mathsf{F}_{\mathsf{k,l}} = \mathsf{j}_\mathsf{k} \otimes \mathsf{u}_\mathsf{l}$, we obtain several relations: $\mathsf{H}_2^{-1}\mathsf{B}^\top = \mathsf{F}_{\mathsf{n_2 n_1}}$, $\mathsf{J}_{\mathsf{n_1 n_2}}\mathsf{F}_{\mathsf{n_2 n_1}} = n_2\mathsf{F}_{\mathsf{n_1 n_1}} = n_2\mathsf{F}_{\mathsf{n_1}}$ and $\mathsf{F}_{\mathsf{n_1}}\mathsf{G}_1{\mathsf{F}_{\mathsf{n_1}}}^\top = g_{11}^1\mathsf{J}_{\mathsf{n_1}}$. Then we obtain the following result:

Proposition 2.1 *The group inverse of the Laplacian of the connection of two networks by a bridge is*

$$(\mathsf{L}')^\dagger = \begin{pmatrix} \mathsf{X}_1 & \mathsf{C} \\ & \\ \mathsf{C}^\top & \mathsf{X}_2 \end{pmatrix},$$

where

$$\begin{aligned} \mathsf{X}_1 &= \mathsf{G}_1 + \frac{n_2}{n}\left(\mathsf{F}_{\mathsf{n}_1}\mathsf{G}_1 + \mathsf{G}_1\mathsf{F}_{\mathsf{n}_1}{}^\top\right) + \frac{n_2^2}{n^2}\left(g_{11}^1 + g_{11}^2 + \frac{1}{a}\right)\mathsf{J}_{\mathsf{n}_1}, \\ \mathsf{X}_2 &= \mathsf{H}_2^{-1} - \frac{1}{n}(\mathsf{J}_{\mathsf{n}_2}\mathsf{H}_2^{-1} + \mathsf{H}_2^{-1}\mathsf{J}_{\mathsf{n}_2}) + \left[\frac{n_1^2}{n^2}g_{11}^1 + \frac{n_2^2}{n^2}\left(g_{11}^2 + \frac{1}{a}\right)\right]\mathsf{J}_{\mathsf{n}_2}, \end{aligned}$$

and

$$\mathsf{C} = -\frac{1}{n}\mathsf{J}_{\mathsf{n}_1\mathsf{n}_2}\mathsf{H}_2^{-1} + \frac{1}{n}\mathsf{G}_1\mathsf{F}_{\mathsf{n}_1\mathsf{n}_2}\mathsf{J}_{\mathsf{n}_2} - \mathsf{G}_1\mathsf{F}_{\mathsf{n}_1\mathsf{n}_2} - \frac{n_1n_2}{n}g_{11}^1\mathsf{J}_{\mathsf{n}_1\mathsf{n}_2} + \frac{n_2^2}{n^2}\left(g_{11}^2 + \frac{1}{a}\right)\mathsf{J}_{\mathsf{n}_1\mathsf{n}_2}.$$

Corollary 2.2 *The Kirchhoff index of the connection of two networks by a bridge is*

$$K(\Gamma) = \frac{n}{n_1}K(\Gamma_1) + \frac{n}{n_2}K(\Gamma_2) + n_1n_2\left(g_{11}^1 + g_{11}^2 + \frac{1}{a}\right).$$

We point out that the Kirchhoff index of a tree coincides with the Wiener Index or mean distance of a tree, and thus we point out that this corollary can be used to compute the mean distance of a tree in terms of the mean distance of some of its branches (see [6]).

Now as a particular example of the connection of two networks by a bridge which is not a tree, we compute the group inverse and the Kirchhoff index of a lollipop graph. That is, if Γ_1 a path of n_1 vertices, P_{n_1}, with constant conductances $c_1 > 0$ and Γ_2 is the complete graph of n_2 vertices, K_{n_2}, with constant conductances c_2, and we connect one of the extremal vertices of the path with any of the vertices of the complete with an edge of conductance $a > 0$, we obtain a lollipop graph. Besides, for any $1 \le i \le j \le n_1$,

$$(\mathsf{G}_1)_{ij} = \frac{1}{6n_1c_1}\left(2n_1^2 + 3n_1 + 1 - 3i - 3j + 3i^2 + 3j^2 - 6n_1j\right).$$

Moreover $\mathsf{G}_2 = \dfrac{1}{n_2^2 c_2}\mathrm{circ}(n_2 - 1, -1, \ldots, -1)$ and

$$\left(\mathsf{H}_2^{-1}\right)_{ij} = \begin{cases} \dfrac{1}{a} & \text{if} \quad i, j = 1, \\[2mm] \dfrac{2a + n_2 c_2}{a n_2 c_2} & \text{if} \quad i = j > 1, \\[2mm] \dfrac{a + n_2 c_2}{a n_2 c_2} & \text{if} \quad i \neq j, \quad i, j > 1. \end{cases}$$

Therefore

$$g_{11}^1 = \frac{1}{6 n_1 c_1}\left(2n_1^2 - 3n_1 + 1\right) \quad \text{and} \quad g_{11}^2 = \frac{n_2 - 1}{n_2^2 c_2},$$

and for any $1 \leq i \leq j \leq n$,

$$\begin{aligned} (\mathsf{X}_1)_{ij} = {} & \frac{1}{c_1 n^2}\left(\frac{n_1^3}{3} + \frac{n_1^2}{2} + \frac{n_1}{6} + \frac{c_1}{c_2}(n_2 - 1) + \frac{c_1 n_2^2}{a} - n_2^2\right) \\ & + \frac{1}{2c_1 n}\left[(2n_2 - 1)(i + j) + (i^2 + j^2)\right] - \frac{j}{c_1}. \end{aligned}$$

$$(\mathsf{X}_2)_{ij} = \frac{1}{n^2}\cdot \begin{cases} \dfrac{n_1^2}{a} + \dfrac{n_2 - 1}{c_2} + \dfrac{2n_1^3 - 3n_1^2 + n_1}{6c_1} & \text{if } \ i, j = 1, \\[2mm] \dfrac{n_1^2}{a} - \dfrac{n_1 + 1}{c_2} + \dfrac{2n_1^3 - 3n_1^2 + n_1}{6c_1} & \text{if } \ 1 = i < j \quad \text{or} \quad 1 = j < i, \\[2mm] \dfrac{n_1^2}{a} + \dfrac{2n_1^2 + n_2^2 + 2n_1 n_2 - n_2}{c_2 n_2} + \dfrac{2n_1^3 - 3n_1^2 + n_1}{6c_1} & \text{if } \ i = j > 1, \\[2mm] \dfrac{n_1^2}{a} + \dfrac{n_1^2 - n_2}{c_2 n_2} + \dfrac{2n_1^3 - 3n_1^2 + n_1}{6c_1} & \text{if } \ i \neq j, \ \ i, j > 1. \end{cases}$$

$$(\mathsf{C})_{ij} = -\frac{n_1 n_2}{a n^2} + \frac{i^2 - (2n_1 + 1)i}{2c_1 n} + \frac{n_1(2n_1^2 + 3n_1 + 1 + 6n_2)}{c_2 n^2} \begin{cases} +\dfrac{n_2 - 1}{c_2 n^2} & \text{if} \quad j = 1. \\[2mm] -\dfrac{n_1 + 1}{c_2 n^2} & \text{if} \quad j > 1. \end{cases}$$

And finally we obtain the Kirchhoff index of the lollipop graph

$$K(\Gamma) = \frac{1}{n}\left[\frac{n_1(n_1-1)(n_1+3n_2+1)}{6c_1} + \frac{(n_2-1)(2n_1+n_2)}{c_2 n_2} + \frac{n_1 n_2}{a}\right].$$

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