On the chromatic number of a subgraph of the Kneser graph

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Abstract

Let $n$ and $k$ be positive integers with $n \geq 2k$. Consider a circle $C$ with $n$ points 1, \ldots, $n$ in clockwise order. The \textit{interlacing graph} $IG_{n,k}$ is the graph with vertices corresponding to $k$-subsets of $[n]$ that do not contain two adjacent points on $C$, and edges between $k$-subsets $P$ and $Q$ if they \textit{interlace}: after removing the points in $P$ from $C$, the points in $Q$ are in different connected components. In this paper we prove that the circular chromatic number of $IG_{n,k}$ is equal to $n/k$, hence the chromatic number is $\lceil n/k \rceil$, and that its independence number is $\binom{n-k-1}{k-1}$.

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1 Introduction

Let $n$ and $k$ be positive integers with $n \geq 2k$. The Kneser graph $KG_{n,k}$, first introduced by Martin Kneser in [4], is the graph with vertex set corresponding to $k$-subsets of $[n] := \{1, \ldots, n\}$, where two vertices are adjacent if the corresponding sets are disjoint. Kneser conjectured that the chromatic number of $KG_{n,k}$ is $n - 2k + 2$. In [5], Lovász proved this conjecture using topological methods. The Schrijver graph $SG_{n,k}$ is the subgraph of $KG_{n,k}$ induced by those vertices that correspond to $k$-subsets of $[n]$ not containing adjacent elements in $[n]$ (here 1 and $n$ are adjacent). In [8], Schrijver showed that $SG_{n,k}$ is a vertex-critical subgraph of $KG_{n,k}$ and also has chromatic number $n - 2k + 2$. (Vertex-critical means that the deletion of any vertex reduces the chromatic number.) Another famous result regarding the Kneser graph is the Erdős-Ko-Rado theorem [2], which says that the maximum size of an independent set of $KG_{n,k}$ is $(\binom{n-1}{k-1})$.

Let $G$ a finite graph, a circular coloring of size $n/k$ is an assignment $\chi : V(G) \rightarrow \mathbb{Z}/n\mathbb{Z}$ such that $\chi(v_1) - \chi(v_2) \in \{k, k + 1, \ldots, -k\}$ mod $n$ if $\{v_1, v_2\}$ is an edge of the graph. The circular chromatic number of a finite graph $G$, $\chi_{\text{circ}}(G)$, can be defined as the minimal rational number $n/k$ for which there exists a circular coloring of size $n/k$. The circular clique $K_{n/k}$ is the graph on vertices $\{0, \ldots, n-1\}$ such that two vertices are adjacent if their distance in $\mathbb{Z}/n\mathbb{Z}$ is larger or equal than $k$. If $G$ is a finite graph then $\lceil \chi_{\text{circ}}(G) \rceil = \chi(G)$ (see, for instance, [10]). In [1], Chen confirmed the conjecture from [3] that $\chi_{\text{circ}}(KG_{n,k}) = \chi(KG_{n,k})$ which was previously known for some cases, such as for even $n$ for the Schrijver graph [6, 9].

In this paper we consider a subgraph of the Kneser graph. If $P$ and $Q$ are two $k$-subsets of $[n]$, $P = \{1 \leq p_1 < \cdots < p_k \leq n\}$ and $Q = \{1 \leq q_1 < \cdots < q_k \leq n\}$, then $P$ and $Q$ are interlacing if either

$$1 \leq p_1 < q_1 < p_2 < q_2 < \cdots < p_k < q_k \leq n$$

or

$$1 \leq q_1 < p_1 < q_2 < p_2 < \cdots < q_k < p_k \leq n.$$ 

By distributing the elements of $[n]$ in clockwise order around a circle we may view $P$ and $Q$ as $k$-polygons with points on the circle. Then $P$ and $Q$ are interlacing if removing the points of $P$ divides the circle into intervals that each contain one point of $Q$. We use this analogy to refer to $k$-subsets in $[n]$ as $k$-polygons, or just polygons when $k$ is understood. We say that a polygon that does not contain two adjacent points on the circle is admissible. The interlacing graph $IG_{n,k}$ is the graph whose vertices correspond to admissible
\( k \)-polygons on \([n]\), and where two vertices are adjacent if the corresponding polygons are interlacing. As non-admissible polygons would give rise to isolated vertices in the interlacing graph, only admissible polygons are considered. Note that \( \text{IG}_{n,k} \) also is a subgraph of the Schrijver graph \( \text{SG}_{n,k} \).

1.1 Main results

Our main result is the following.

**Theorem 1.1** *The circular chromatic number of \( \text{IG}_{n,k} \) is equal to \( n/k \).*

Thus, we obtain that:

**Corollary 1.2** *The chromatic number of \( \text{IG}_{n,k} \) is equal to \( \lceil n/k \rceil \).*

We also determine the independence number of the interlacing graph.

**Proposition 1.3** *The independence number of \( \text{IG}_{n,k} \) is \( \binom{n-k-1}{k-1} \).*

To prove Theorem 1.1 we find a circular clique \( K_{n/k} \) as a subgraph in \( \text{IG}_{n,k} \). Afterwards we find a circular coloring \( \chi \) of size \( n/k \), or a graph homomorphism from \( \text{IG}_{n,k} \) to \( K_{n/k} \). The vertex color classes induced by the coloring \( \chi \) can be naturally grouped into stable sets of maximum (or almost maximum) size.

The interlacing graph has connections with triangulations of cyclic polytopes. Note that if \( k = 2 \), two non-interlacing polygons on \([n]\) are just two non-crossing lines between vertices of an \( n \)-polygon. A maximal set of pairwise non-interlacing polygons is a triangulation. In [7], Oppermann and Thomas generalized this observation to higher dimensions: the set of triangulations of the cyclic polytope with \( n \) vertices in dimension \( 2k - 2 \), are in bijection with the independent sets of admissible polygons in \( \text{IG}_{n,k} \) of maximal size. (The cyclic polytope \( C(n, 2k-2) \) is the convex hull of \( n \) distinct points in \( \mathbb{R}^{2k-2} \) that are obtained as evaluations of the curve defined by \( P(x) = (x, x^2, \ldots, x^{2k-2}) \), which is called the moment curve.) In particular, the chromatic number of \( \text{IG}_{n,k} \) gives the minimal size of a partition of the \((k-1)\)-dimensional internal simplices of \( C(n, 2k-2) \) (see [7]) in which no two simplices in each part internally intersect. The proof of Proposition 1.3 exploits this connection and follows the framework developed in [7].

2 Graph parameters of \( \text{IG}_{n,k} \)

2.1 *The independence number*

The proof of Proposition 1.3 uses a standard counting argument.
Lemma 2.1 The number of admissible $k$-polygons on $[n]$ containing a specific point on the circle is $\binom{n-k-1}{k-1}$. In particular, the number of vertices of $IG_{n,k}$ is $\frac{n}{k}\binom{n-k-1}{k-1}$.

The combination of Lemma 2.1 with the techniques and arguments appearing in [7] show Proposition 1.3.

2.2 Lower bound for the circular chromatic number

The following lemma gives the lower bound to the circular chromatic number for the interlacing graph.

Lemma 2.2 Let $n, k$ be positive integers, $n \geq 2k$, and let $n', k'$ be coprime positive integers such that $n'/k' = n/k$. The subgraph of $IG_{n,k}$ induced by the $n'$ polygons $\{P_j\}_{j \in [0,n'-1]}$, $P_j = \{j+n, j+[n/k], j+[2n/k], \ldots, j+[in/k], \ldots, j+[(k-1)n/k]\}$ is a circular clique $K_{n'/k'}$.

Lemma 2.2 follows by coloring $P_j$ with color $jk'$ in $\mathbb{Z}/n'\mathbb{Z}$ and observing that the edges are precisely between the claimed polygons (see, for instance, Lemma 2.5).

2.3 A circular coloring matching the lower bound

We show Theorem 1.1 using the following auxiliary technical lemma and its consequences.

Lemma 2.3 Let $y_1, \ldots, y_k \in \mathbb{R}_{\geq 0}$ and let $\sum_{i=1}^{k} y_i = z$. Then there exists a $j_0 \in [k]$ such that for all $m \in [k]$, $\sum_{i=j_0}^{j_0+m-1} y_i \geq mz/k$, where the indices are taken modulo $k$. Moreover, either there exists an $m' \in [k]$ for which $\sum_{i=j_0}^{j_0+m'-1} y_i > m'z/k$, or $y_i = z/k$ for each $i \in [k]$.

A pigeonhole argument combined with an induction on $k$ shows Lemma 2.3.

For a $k$-polygon $P = \{1 \leq y_1 < \ldots < y_k \leq n\}$ on the circle with points $1, \ldots, n$ in clockwise order, define the $k$-tuple of distances between the consecutive points $s(P) := (y_2 - y_1, y_3 - y_2, \ldots, y_1 - y_k + n) \in \mathbb{Z}^k_{\geq 1}$. We call $s(P)$ the shape of $P$. We say that the $k$-polygon with points $y_1 + 1, \ldots, y_k + 1$ is obtained from $P$ by a clockwise rotation of 1 (the addition is modulo $n$). For $i \geq 0$, the $k$-polygon obtained by rotating clockwise $i$ times is denoted by $\rho_i(P)$. 
The following corollary of Lemma 2.3 shows that every polygon can be rotated to contain \( n \) and such that the \( i \)-th point on the polygon is at a distance larger or equal than \( \lceil in/k \rceil \) to the point \( n \) (in the counterclockwise direction).

**Corollary 2.4** Let \( P = \{1 \leq y_1 < \ldots < y_k \leq n\} \) be a \( k \)-polygon and write \( s(P) = (d_1, \ldots, d_k) \) for the shape of \( P \). Then there is a \( j_0 \in [k] \) such that for \( i' = n - y_{j_0} \) we have

\[
n \in \rho_{i'}(P) \quad \text{and} \quad |\rho_{i'}(P) \cap \{1, \ldots, \lfloor mn/k \rfloor\}| \leq m \quad \text{for all} \quad m \in [k].
\]

Additionally, for \( m \in [k] \) we have

\[
|\rho_{i'}(P) \cap \{1, \ldots, \lfloor mn/k \rfloor\}| = m \iff \lfloor mn/k \rfloor = mn/k \quad \text{and} \quad \sum_{i=j_0}^{j_0+m-1} d_i = mn/k.
\]

For a vector \( d = (d_1, \ldots, d_k) \in \mathbb{Z}^k_{\geq 2} \) with \( \sum_{i=1}^{k} d_i = n \), let \( P_d^n \) be the \( k \)-polygon with \( s(P_d^n) = d \) and containing the point \( n \). Note that \( P_d^n \) is admissible. The set of \( k \)-polygons of the form \( P_d^n \) (for some \( d \) as above) is an independent set in \( IG_{n,k} \). Define

\[
\mathcal{L}_{n,k} := \{ P_d^n \mid d = (d_1, \ldots, d_k) \in \mathbb{Z}^k_{\geq 2} \quad \text{and} \quad \sum_{i=1}^{t} d_i \geq tn/k \quad \text{for all} \quad t \in [k]\}.
\]

The next lemma summarizes the main properties of the polygons in \( \mathcal{L}_{n,k} \).

**Lemma 2.5** For any \( j, i \in [0, n] \), \( \{\rho_j(\mathcal{L}_{n,k}), \rho_j+[in/k]_{\mathcal{L}_{n,k}}\} \) and \( \{\rho_j(\mathcal{L}_{n,k}), \rho_j+[in/k]_{\mathcal{L}_{n,k}}\} \) are independent sets, where \( \rho_i(\mathcal{L}_{n,k}) = \{\rho_i(Q) \mid Q \in \mathcal{L}_{n,k}\} \).

Indeed, if \( P \in \mathcal{L}_{n,k} \) and \( Q \in \rho_{[in/k]}(\mathcal{L}_{n,k}) \) (resp. \( Q \in \rho_{[in/k]}(\mathcal{L}_{n,k}) \)), then between \( n \) and \( \lceil in/k \rceil \) (resp. \( \lceil in/k \rceil \)) \( Q \) contains \( i + 1 \) point. On the other side, either \( P \) contains at most \( i \) point between \( n \) and \( \lceil in/k \rceil \) (resp. \( \lfloor in/k \rfloor \)), or both share the point \( in/k \) if \( \lfloor in/k \rfloor = \lceil in/k \rceil \) (resp. they share \( \lfloor in/k \rfloor \)) if \( P \) also contains \( i + 1 \) in such interval. The same argument applies if both sets \( \mathcal{L}_{n,k} \) and \( \rho_{[in/k]}(\mathcal{L}_{n,k}) \) (or \( \rho_{[in/k]}(\mathcal{L}_{n,k}) \)) are rotated by \( \rho_j \).

The remaining part of the argument to show Theorem 1.1 can now be sketched. Any polygon is the rotation of a polygon in \( \mathcal{L}_{n,k} \), by Corollary 2.4. The map \( \chi \) colors the polygon \( P \) with \( i \cdot k \in \mathbb{Z}/n\mathbb{Z} \) if \( i \) is the minimal index in \([0, n - 1]\) such that \( P \in \rho_i(\mathcal{L}_{n,k}) \). Lemma 2.5 shows that the coloring \( \chi \) is indeed a circular coloring with \( n/k \) colors.
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References


