Edges incident with a vertex of degree greater than four and a lower bound on the number of contractible edges in a 4-connected graph

Shunsuke Nakamura¹, Yoshimi Egawa

Department of Applied Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Keiko Kotani²

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Abstract

In this paper, we prove that the number of 4-contractible edges (edges that after contraction do not change the connectivity of the initial graph) of a 4-connected graph $G$ is at least $(1/28) \sum_{x \in V_{\geq 5}(G)} \deg_G(x)$, where $V_{\geq 5}(G)$ denotes the set of those vertices of $G$ which have degree greater than or equal to 5.

This is the refinement of the result proved by Ando et al. [On the number of 4-contractible edges in 4-connected graphs, J. Combin. Theory Ser. B 99 (2009) 97–109].

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¹ Email: 1417701@ed.tus.ac.jp
² Email: kkotani@rs.kagu.tus.ac.jp
1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the reader to [4].

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $N_G(x)$ denotes the neighborhood of $x$ and $\deg_G(x)$ denotes the degree of $x$; thus $\deg_G(x) = |N_G(x)|$. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of $e$. The complete graph of order $n$ is denoted by $K_n$. The complete bipartite graph $K_{1,n}$ with partite sets of cardinalities 1 and $n$ is called a star. For a graph $H$, let $nH$ denote the graph with $n$ components, each isomorphic to $H$. For an integer $i \geq 0$, we let $V_i(G)$ denote the set of vertices $x$ of $G$ with $\deg_G(x) = i$ and we let $V \geq i(G) = \bigcup_{j \geq i} V_j(G)$. A subset $S$ of $V(G)$ is called a cutset if $G - S$ is disconnected. For an integer $k \geq 1$, we say that $G$ is $k$-connected if $|V(G)| \geq k + 1$ and $G$ has no $(k - 1)$-cutset.

Let $G$ be a 4-connected graph. For $e \in E(G)$, we let $G/e$ denote the graph obtained from $G$ by contracting $e$ into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that $e$ is 4-contractible or 4-noncontractible according as $G/e$ is 4-connected or not. A 4-noncontractible edge $e = ab$ is said to be trivially 4-noncontractible if there exists a vertex $z$ of degree 4 such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_t(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively.

The following characterization of 4-connected graphs with $E_c(G) = \emptyset$ was obtained by Fontet and independently by Martinov.

**Theorem A (Fontet [7]; Martinov [10])** Let $G$ be a 4-connected graph of order $n$ with $E_c(G) = \emptyset$. Then one of the following holds:

1. $G$ is the square of the cycle of order $n$; i.e., we can write $V(G) = \{v_1, v_2, \ldots, v_n\}$ so that $E(G) = \{v_iv_j \mid i - j \in \{\pm 1, \pm 2\} \text{ (mod } n)\}$; or
2. there exists a 3-regular graph $H$ such that $G$ is the line graph of $H$.

In view of Theorem A, it is natural to expect that one can estimate $|E_c(G)|$ in terms of degrees of vertices of $G$, and also in terms of the number of edges of $G$ not contained in a triangle. Along this line, the following results have been obtained.

**Theorem B (Ando, Egawa, Kawababayashi and Kriesell [3])** If $G$ is a 4-connected graph, then $|E_c(G)| \geq (1/68) \sum_{u \in V(G)} (\deg_G(u) - 4)$. 

Theorem C (Ando and Egawa [1]) If $G$ is a 4-connected graph, then $|E_c(G)| \geq |V_{\geq 5}(G)|$.

Further we let $\hat{E}(G)$ denote the set of those edges of $G$ which are not contained in a triangle. Let $\hat{V}$ denote the set of those vertices of $G$ which are incident with an edge in $\hat{E}(G) \cap E_n(G)$, and let $\hat{G}$ denote the subgraph of $G$ induced by the edge set $\hat{E}(G) \cap E_n(G)$; that is to say, $\hat{V} = \bigcup_{e \in \hat{E}(G) \cap E_n(G)} V(e)$ and $\hat{G} = (\hat{V}, \hat{E}(G) \cap E_n(G))$. Finally we let $Y^*$ denote the graph of order 6 defined by $V(Y^*) = \{w, z\} \cup \{v_i \mid 1 \leq i \leq 4\}$, $E(Y^*) = \{wz, v_1w, v_2w, v_3z, v_4z\}$.

Theorem D (Ando and Egawa [2]) Let $G$ be a 4-connected graph, and suppose that $|\hat{E}(G)| \geq 15$. Then $|E_c(G)| \geq \left( |\hat{E}(G)| + 8 \right) / 4$.

In Theorems C and D, the lower bound on $|E_c(G)|$ is best possible. However, the bound 15 on $|\hat{E}(G)|$ in the assumption of Theorem D is not best possible. In fact, the following theorem concerning the refinements of Theorem D has already been proved.

Theorem E (Egawa et al. [5,6]; Kotani et al. [9]; Nakamura [11]) Let $G$ be a 4-connected graph, and suppose that $1 \leq |\hat{E}(G)| \leq 14$. Then $|E_c(G)| \geq \left( |\hat{E}(G)| + 8 \right) / 4$ unless one of the following holds:

1. $|\hat{E}(G)| = 1$ and $\hat{G} = K_2$;
2. $|\hat{E}(G)| = 2$ and $\hat{G} = \emptyset$;
3. $|\hat{E}(G)| = 3$ and $\hat{G} = K_2$;
4. $|\hat{E}(G)| = 4$ and $\hat{G} = 2K_2$;
5. $|\hat{E}(G)| = 5$ and $\hat{G} = 2K_2$ or $K_{1,2}$;
6. $|\hat{E}(G)| = 6$ and $\hat{G} = 3K_2$; or
7. $|\hat{E}(G)| = 9$ and $\hat{G} = Y^*$.

In Theorem B, the coefficient $1/68$ seems far from best possible. The purpose of this paper is to prove the following theorem which is the refinement of Theorem B.

Theorem 1 If $G$ is a 4-connected graph, then

$$|E_c(G)| \geq \frac{1}{28} \sum_{u \in V_{\geq 5}(G)} \deg_G(u).$$

The coefficient $1/28$ in Theorem 1 still seems not to be best possible. However we construct examples showing that the coefficient of Theorem 1 is
at most 1/13.

The organization of this paper is as follows. In Section 2, we introduce a known result proved in [8] and introduce some lemmas for the proof of Theorem 1. Finally, we prove Theorem 1 in Section 3.

2 Preliminaries

Throughout the rest of this paper, we let $G$ be a 4-connected graph. Let $L$ be the set of edges $e$ such that both endvertices of $e$ have degree 4, and let $F = E(G) - E(G - L)$. Also let $V(G)$ denote the set of those vertices of $G$ which are incident with an edge in $F$, and let $G$ denote the spanning subgraph of $G$ with edge set $F$; that is, $V(G) = \bigcup_{e \in F} V(e)$ and $G = (V(G), F)$.

Set $L = \{(S, A) \mid S$ is a 4-cutset, $A$ is the union of the vertex set of some components of $G - S$, $\emptyset \neq A \neq V(G) - S\}$.

Now take $(S_1, A_1), \ldots, (S_k, A_k) \in L$ so that for each $e \in F$, there exists $S_i$ such that $V(e) \subseteq S_i$. We choose $(S_1, A_1), \ldots, (S_k, A_k)$ so that $k$ is minimum and so that $(|A_1|, \ldots, |A_k|)$ is lexicographically minimum, subject to the condition that $k$ is minimum. Set $S = \{S_1, \ldots, S_k\}$.

For two distinct 4-cutset $S, T \in \mathcal{S}$, we say that $S$ crosses $T$ if $S$ intersects with every component of $G - T$. Furthermore, we call $\mathcal{S}$ is cross free if any two members of $\mathcal{S}$ do not cross. The following lemma plays an important role in the proof of Theorem 1.

Lemma 2.1 (Kotani and Nakamura [8]) Suppose that $|V(G)| \geq 9$ and some two members of $\mathcal{S}$ cross. Then there exists a 4-connected graph $G'$ such that $|V(G')| = |V(G)| - 2$ and

$$|E_c(G)| - |E_c(G')| \geq \max \left\{ 1, \frac{1}{10} \left( \sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x) \right) \right\}.$$ 

We introduce two lemmas for the proof of Theorem 1. In order to introduce the first result, we set $R = \{(u, a) \mid ua \in E(G) - F, u \in V_{\geq 5}(G)\}$ and $Q = \{(x, y) \mid xy \in E_c(G)\}$. Then the following lemma holds.

Lemma 2.2 Suppose that $\mathcal{S}$ is cross free. Then $|R| \leq 4|Q|$.

We also set $J = \{(u, a) \mid ua \in F, u \in V_{\geq 5}(G)\}$. Now we introduce the second result which will be used in the proof of Theorem 1.
Lemma 2.3 Suppose that $S$ is cross free. Then $|J| \leq 10|Q|$.

3 Proof of Theorem 1

In this section, we prove Theorem 1. If $G$ is 4-regular, then $\sum_{x \in V_{\geq 5}(G)} \deg_G(x) = 0$, and hence the desired inequality holds immediately. Thus we may assume that $G$ is not 4-regular, thus $|V_{\geq 5}(G)| \geq 1$. By way of contradiction, we suppose that $|E_c(G)| < (1/28) \sum_{x \in V_{\geq 5}(G)} \deg_G(x)$, thus $\sum_{x \in V_{\geq 5}(G)} \deg_G(x) > 28|E_c(G)|$. We may assume that we have chosen $G$ such that $|V(G)|$ is as small as possible. Then the following claim holds.

Claim 3.1 $|V(G)| \geq 9$.

Proof. Suppose that $|V(G)| \leq 8$. Set $b(G) := |E(G)| - 2|V(G)|$. Then $b(G) = (1/2) \sum_{x \in V_{\geq 5}(G)} \deg_G(x) - 2|V_{\geq 5}(G)| > 14|E_c(G)| - 2|V_{\geq 5}(G)|$. Since $G$ is a 4-connected graph such that $G$ is not 4-regular, we have $6 \leq |V(G)| \leq 8$, and hence $b(G) = |E(G)| - 2|V(G)| \leq \max\{(15-12), (21-14), (28-16)\} = 12$. By Theorem C, $b(G) > 14|E_c(G)| - 2|V_{\geq 5}(G)| \geq 14|V_{\geq 5}(G)| - 2|V_{\geq 5}(G)| = 12|V_{\geq 5}(G)| \geq 12$, which contradicts $b(G) \leq 12$.

Let $S$ be as in Section 2. By making use of Claim 3.1, we prove the following claim.

Claim 3.2 $S$ is cross free.

Proof. Suppose that some two members of $S$ cross. By Lemma 2.1 and Claim 3.1, there exists a 4-connected graph $G'$ such that $|V'(G')| = |V(G)| - 2$ and $|E_c(G') - |E_c(G')| \geq \max\{1, (1/10)(\sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x))\}$. Since $|V'(G')| < |V(G)|$, we have $|E_c(G')| \geq (1/28) \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x)$. Thus

$$1 \leq |E_c(G)| - |E_c(G')| < \frac{1}{28} \left( \sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x) \right) < \frac{1}{10} \left( \sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x) \right),$$

which is a contradiction.

We are now in a position to complete the proof of Theorem 1. Note that $|R| + |J| = |\{ (x, y) \mid xy \in E(G), x \in V_{\geq 5}(G) \}| = \sum_{x \in V_{\geq 5}(G)} \deg_G(x)$. It follows
from Lemmas 2.2, 2.3 and Claim 3.2 that
\[
\sum_{x \in V_{\geq 5}(G)} \deg_G(x) = |R| + |J| \leq 4|Q| + 10|Q| = 14|Q| = 28|E_c(G)|,
\]
which contradicts the assumption that \(|E_c(G)| < (1/28) \sum_{x \in V_{\geq 5}(G)} \deg_G(x)\).

This completes the proof of Theorem 1. \qed

References


