1 Introduction

A group $K$ \textit{virtually algebraically fibers} if there is a finite index subgroup $K'$ admitting a surjective homomorphism $K' \to \mathbb{Z}$ with finitely generated kernel. This notion arises from topology: a 3-manifold $M$ is virtually a surface bundle over a circle precisely when the fundamental group of $M$ virtually algebraically fibers (see the result of Stallings [10]). A \textit{Right-Angled Coxeter group} (RACG) $K$ is a group given by a presentation of the form

$$\langle x_1, x_2, \ldots, x_n \mid x_i^2, [x_i, x_j]^{\sigma_{ij}} : 1 \leq i < j \leq n \rangle$$

where $\sigma_{ij} \in \{0, 1\}$ for each $1 \leq i < j \leq n$. One can encode this information with a graph $\Gamma_K$ whose vertices are the generators $x_1, \ldots, x_n$ and $x_i \sim x_j$ if
and only if $\sigma_{ij} = 1$. Conversely given a graph $G$ on $n$ vertices, we will denote the corresponding RACG by $K(G)$.

Random Coxeter groups have been of heightened recent interest, see for instance Charney and Farber [4], Davis and Kahle [5], and Behrstock, Falgas-Ravry, Hagen, and Susse [1].

Recently, Jankiewicz, Norin, and Wise [8] developed a framework to show virtual fibering of a RACG using Betsvina-Brady Morse theory [3] and ultimately translated the virtual fibering problem for $K$ into a combinatorial game on the graph $\Gamma_K$. The method was successful on many special cases and also allowed them to construct examples where Betsvina-Brady cannot be applied to find a virtual algebraic fibering.

A natural question to consider is whether this approach is successful for a ‘generic’ RACG, i.e., given a probability measure $\mu_n$ on the set of RACG’s of rank at most $n$, is it true that a.a.s. as $n \to \infty$, a group sampled from $\mu_n$ virtually algebraically fibers. This question is also considered in [8], specifically they consider sampling $\Gamma_K$ from the Erdős-Rényi random graph model $G(n, p)$ and they prove the following result:

**Theorem 1.1 (Jankiewicz-Norin-Wise)** Assume that
\[
\frac{(2\log n)^{\frac{1}{2}} + \omega(n)}{n^2} \leq p < 1 - \omega(n^{-2}),
\]
and let $G$ be sampled from $G(n, p)$. Then, asymptotically almost surely, the associated Right-Angled Coxeter group $K(G)$ virtually algebraically fibers.

In this paper we extend this result to the smallest possible range of $p$, in fact we prove a hitting time type result. Namely we show that as soon as $\Gamma_K$ has minimum degree 2 then a.a.s. $K$ virtually algebraically fibers.

**Theorem 1.2** Let $G_0, G_1, \ldots, G_{\binom{n}{2}}$ denote the random graph graph process on $n$ vertices where $G_{i+1} = G_i \cup \{e_i\}$ and $e_i$ is picked uniformly at random from the non-edges of $G_i$. Let $T = \min\{t : \delta(G_t) = 2\}$, then a.a.s. the random graph process is such that $K(G_m)$ virtually algebraically fibers if and only if $T \leq m < \binom{n}{2}$. In particular for any $p$ satisfying
\[
\frac{\log n + \log \log n + \omega(n)}{n} \leq p < 1 - \omega(n^{-2})
\]
and $G \ G(n, p)$, the random Right-Angled Coxeter group $K(G)$ virtually algebraically fibers a.a.s.
2 The combinatorial game

In this section we follow the definitions in [8] to present the combinatorial game introduced in [8] used to construct virtual algebraic fiberings of Right-Angled Coxeter groups.

**Definition 2.1** Let $G = (V,E)$ be a graph. We say that a subset $S \subseteq V$ is a legal state if both $S$ and $V \setminus S$ are non-empty connected subsets of $V$, i.e., the corresponding induced graphs are connected and non-empty.

**Definition 2.2** For each $v \in V$, a move at $v$ is a set $M_v \subseteq V$ satisfying the following:
- $v \in M_v$.
- $N(v) \cap M_v = \emptyset$.

Let $\mathcal{M} = \{M_v : v \in V\}$ denote a set of moves.

We will identify subsets of $V$ as elements of $\mathbb{Z}_2^V$ in the obvious way. Thus each state and each move correspond to elements of $\mathbb{Z}_2^V$ and we will think of moves acting on states via group multiplication (or addition in this case).

**Definition 2.3** For a graph $G$, a state $S \subseteq V(G)$, and a set of moves $\mathcal{M} = \{M_v : v \in V\}$, the triple $(G,S,\mathcal{M})$ is a legal system if for any element $g \in \langle \mathcal{M} \rangle$, $g(S)$ is a legal state of $G$.

**Theorem 2.4** ([8]) Let $(G,S,\mathcal{M})$ be a legal system, then the RACG $K(G)$ must virtually algebraically fiber.

To elucidate the notion of a legal system, let us look at some toy examples (see Figure 2) and ask whether each of these graphs contains a legal system.

**Example 2.5** Let $G = (V,E)$ be a graph with three vertices $V = \{v,u_1,u_2\}$ and two edges $E = \{\{v,u_1\},\{v,u_2\}\}$. We show that $G$ has a legal system. Our initial legal state will be $S = \{u_1\}$. For our set of moves we choose
\[ M_v = \{v\} \] (note that this is the only possible choice for the move at \( v \)), \[ M_{u_1} = M_{u_2} = \{u_1, u_2\} \]. Then the group generated by the moves of the graph, written as a collection of sets, is \( \langle \mathcal{M} \rangle = \{\{v\}, \{u_1, u_2\}, \{v, u_1, u_2\}, \emptyset\} \). Hence, for any element \( g \in \langle \mathcal{M} \rangle \), \( g(S) \) is either a set of the form \( \{u_i\} \) or \( \{v, u_i\} \), for \( i = 1, 2 \), and in any case a legal state. Thus, \( (G, S, \mathcal{M}) \) is a legal system.

The graph in Example 2.5 is unique in the sense that it is the only graph with a vertex of degree 1 on at least 3 vertices which contains a legal system.

Next, we look at an example of a graph without a legal system. We proceed by exhaustion.

**Example 2.6** Let \( G \) be the bowtie graph on 4 vertices. Assume by contradiction that \( (G, S, \mathcal{M}) \) is a legal system. Since \( v \) is connected to all other vertices in the graph, we must have \( M_v = \{v\} \). For the same reason, \( v \) cannot belong to any other move apart from \( M_v \). Hence, we can assume without loss of generality that \( v \notin S \). Since \( S \) is a connected subset of \( V \), we can again assume without loss of generality that \( S = \{u_1\} \) or \( S = \{u_1, u_2\} \).

In the latter case, \( M_{w_i} = \{u_1, u_2, w_i\} \) for \( i = 1, 2 \), because by the definition of a move, it must be the case that \( \{w_i\} \subseteq M_{w_i} \subseteq \{w_i, u_1, u_2\} \), and if \( u_1 \) or \( u_2 \) would not belong to \( M_{w_i} \), then \( M_{w_i}S \) would not be a legal state. But then the set \( \{w_1, w_2\} \in \langle \mathcal{M} \rangle \), and \( \{w_1, w_2\}S = \{w_1, w_2, u_1, u_2\} \) is not a legal state. In the former case, from similar consideration, it must be the case that \( M_{w_i} = \{w_i, u_1\} \) for \( i = 1, 2 \), but then again \( \{w_1, w_2\} \in \langle \mathcal{M} \rangle \), and \( \{w_1, w_2\}S = \{w_1, w_2, u_1\} \) is not a legal state.

3 **Quick note on method.**

The first ingredient of the proof is to pick the colour classes of vertices as the moves and to choose the starting set \( S \) uniformly at random (independently of the graph). This observation allows us already get close to the threshold but not all the way: for instance an obvious obstruction is that at the target density there will be bounded vertices of degree at most \( C \) with some probability bounded away from 0 and thus with some probability bounded away from 0 these will be isolated in \( S \).

The second ingredient then is to show that one can modify the original random selection of \( S \) and the moves to accommodate for the obstructions.

Finally, in order to prove a hitting time result, we show that any graph that deterministically satisfies certain pseudorandom properties must accept a legal system. The task then is to show that at the hitting time \( T \), \( G_T \) satisfies said pseudorandom properties with high probability.
References


