Graph Operations Preserving W_2-Property

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Abstract

A graph is *well-covered* if all its maximal independent sets are of the same size (Plummer, 1970). A graph G belongs to class $\mathbf{W}_n$ if every n pairwise disjoint independent sets in G are included in n pairwise disjoint maximum independent sets (Staples, 1975). Clearly, $\mathbf{W}_1$ is the family of all well-covered graphs. Staples showed a number of ways to build graphs in $\mathbf{W}_n$, using graphs from $\mathbf{W}_n$ or $\mathbf{W}_{n+1}$. In this paper, we construct some more infinite subfamilies of the class $\mathbf{W_2}$ by means of corona, join, and rooted product of graphs.

Keywords: independent set, well-covered graph, class $\mathbf{W}_2$, shedding vertex, corona of graphs, graph join, rooted product of graphs.

1 Introduction

Throughout this paper, G is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V(G) \neq \emptyset$ and edge set $E(G)$.

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The *neighborhood* $N(v)$ of $v \in V(G)$ is the set $\{w : w \in V(G)$ *and* $vw \in E(G)\}$, while $N[v] = N(v) \cup \{v\}$. The *neighborhood* $N(A)$ of a set $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$. We may also use $N_G(v), N_G[v], N_G(A)$ and $N_G[A]$, when referring to neighborhoods in a graph G. We let C_n, K_n, P_n denote respectively, the cycle on $n \geq 3$ vertices, the complete graph on $n \geq 1$ vertices, the path on $n \geq 1$ vertices.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$. Let $\Omega(G) = \{S \in \mathrm{Ind}(G) : |S| = \alpha(G)\}$.

A graph is *well-covered* if all its maximal independent sets are of the same cardinality [6]. A graph G belongs to the class $\mathbf{W}_n, n \geq 1$, if every n pairwise disjoint independent sets in G are included in n pairwise disjoint maximum independent sets [7]. Clearly, $\mathbf{W}_1 \supseteq \mathbf{W}_2 \supseteq \mathbf{W}_3 \supseteq \cdots$, where $\mathbf{W}_1$ is the family of all well-covered graphs.

Theorem 1.1 *[3,4] Let G be a graph without isolated vertices. Then G is in the class $\mathbf{W}_2$ if and only if for every non-maximum independent set A in G and $v \notin A$, there exists some $S \in \Omega(G)$ such that $A \subset S$ and $v \notin S$.*

A vertex v is *shedding* ($v \in \mathrm{Shed}(G)$) if for every $S \in \mathrm{Ind}(G - N[v])$, there is some $u \in N(v)$ such that $S \cup \{u\} \in \mathrm{Ind}(G)$ [10]. Clearly, no isolated vertex may be a shedding vertex, and no $G \in \mathbf{W}_2$ may have isolated vertices.

Theorem 1.2 *[3,4] Let G be a well-covered graph without isolated vertices. Then G belongs to the class $\mathbf{W_2}$ if and only if $\mathrm{Shed}(G) = V(G)$.*

Several ways to build graphs belonging to $\mathbf{W}_n$ are presented in [5,7,8].

In this paper, we describe how to create some more infinite subfamilies of $\mathbf{W_2}$, by means of corona, join, and rooted product.

2 Results

Let $\mathcal{H} = \{H_v : v \in V(G)\}$. The corona $G \circ \mathcal{H}$ is the disjoint union of G and $H_v, v \in V(G)$, with additional edges joining each vertex $v \in V(G)$ to all the vertices of H_v. If $H_v = H$ for every $v \in V(G)$, then we write $G \circ H$ [1].

Proposition 2.1 *[9] The corona $G \circ \mathcal{H}$ of G and $\mathcal{H} = \{H_v : v \in V(G)\}$ is well-covered if and only if each $H_v \in \mathcal{H}$ is a complete graph.*

For example, all the graphs in Figure 1 are of the form $G \circ \mathcal{H}$, but only G_1 is not well-covered, while $G_3 \in \mathbf{W}_2$.

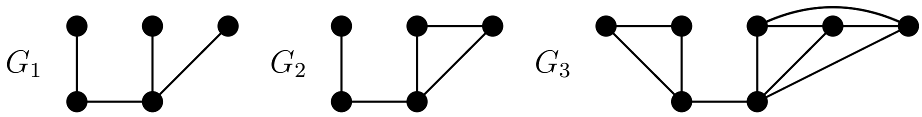

Fig. 1. $G_1 = P_2 \circ \{K_1, 2K_1\}$, $G_2 = P_2 \circ \{K_1, K_2\}$, $G_3 = P_2 \circ \{K_2, K_3\}$.

Proposition 2.2 *Let $L = G \circ \mathcal{H}$, where $\mathcal{H} = \{H_v : v \in V(G)\}$ and G is an arbitrary graph. Then L belongs to $\mathbf{W}_2$ if and only if each $H_v \in \mathcal{H}$ is a complete graph of order at least two, for every non-isolated vertex v, while for each isolated vertex u, its corresponding H_u may be any complete graph.*

Proof. Suppose that $L \in \mathbf{W}_2$. Then L is well-covered, and therefore, each $H_v \in \mathcal{H}$ is a complete graph on at least one vertex, by Proposition 2.1. Let us assume that for some non-isolated vertex $a \in V(G)$ its corresponding graph $H_a = K_1 = (\{b\}, \emptyset)$. Let $c \in N_G(a)$ and B be a non-maximum independent set in L containing c. Since $\alpha(L) = |V(G)|$, it follows that every maximum independent set S of L that includes B must contain the vertex b as well. In other words, L could not be in $\mathbf{W}_2$, according to Theorem 1.1. Therefore, each $H_v \in \mathcal{H}$ must be a complete graph on at least two vertices.

Conversely, if each $H_v \in \mathcal{H}$ is a complete graph on at least two vertices, then L is well-covered, by Proposition 2.1. Let A be a non-maximum independent set in L, and some vertex $b \notin A$. Since L is well-covered, there is some maximum independent set S_1 in L such that $A \subset S_1$. If $b \in S_1$, let $a \in N_L(b) - V(G)$. Hence, $S_2 = S_1 \cup \{a\} - \{b\} \in \Omega(L)$ and $A \subset S_2$. In other words, there is a maximum independent set in L, namely $S \in \{S_1, S_2\}$, such that $A \subset S$ and $b \notin S$. Therefore, by Theorem 1.1, we get that $L \in \mathbf{W}_2$. If v is isolated in G, then even $H_v = K_1$ ensures $L \in \mathbf{W}_2$. □

Corollary 2.3 *If $E(G) \neq \emptyset$, then $G \circ K_p \in \mathbf{W}_2$ if and only if $p \geq 2$.*

If $G_1, G_2, ..., G_p$ are pairwise vertex disjoint graphs, then their *join* is the graph $G = \sum_{1 \leq i \leq p} G_i$ with $V(G) = \bigcup_{1 \leq i \leq p} V(G_i)$ and $E(G) = \bigcup_{1 \leq i \leq p} E(G_i) \cup \{xy : x \in V(G_i), y \in V(G_j), 1 \leq i < j \leq p\}$.

Proposition 2.4 *[9] The graph $G_1 + G_2 + \cdots + G_p$ is well-covered if and only if each G_k is well-covered and $\alpha(G_i) = \alpha(G_j)$ for every $i, j \in \{1, 2, ..., p\}$.*

Lemma 2.5 $\text{Shed}(G_1 + G_2) = \text{Shed}(G_1) \cup \text{Shed}(G_2)$.

Proposition 2.6 *The graph $G = G_1 + G_2 + \cdots + G_p$ belongs to $\mathbf{W}_2$ if and only if each $G_k \in \mathbf{W}_2$ and $\alpha(G_i) = \alpha(G_j)$ for every $i, j \in \{1, 2, ..., p\}$.*

Proof. Suppose that $G \in \mathbf{W}_2$. Let A be a non-maximum independent set in some G_k and $v \in V(G_k) - A$. By Theorem 1.1, there exists some $S \in \Omega(G)$

such that $A \subset S$ and $v \notin S$. Since each vertex of A is joined by an edge to every vertex of $G_i, i \neq k$, we get that $S \in \text{Ind}(G_k)$. Since $S \in \Omega(G)$, we conclude that $S \in \Omega(G_k)$. Therefore, every G_k must be in W_2, in accordance with Theorem 1.1. By Proposition 2.4, we infer that, necessarily, each G_k must be well-covered, and $\alpha(G_i) = \alpha(G_j)$ for every $1 \leq i < j \leq p$.

Let us prove the converse. Since $\alpha(G_i) = \alpha(G_j)$ for every $i, j \in \{1, 2, ..., p\}$, and each $G_k, 1 \leq k \leq p$, is well-covered, Proposition 2.4 implies that G is well-covered as well. According to Lemma 2.5 and Theorem 1.2 we obtain $\text{Shed}(G) = \bigcup_{1 \leq i \leq p} \text{Shed}(G_i) = \bigcup_{1 \leq i \leq p} V(G_i) = V(G)$. In conclusion, Theorem 1.2 tells us that $G \in \mathbf{W}_2$, since G is well-covered and $\text{Shed}(G) = V(G)$.□

Corollary 2.7 *[8] If $G_1, G_2 \in \mathbf{W}_2$ and $\alpha(G_1) = \alpha(G_2)$, then $G_1 + G_2 \in \mathbf{W}_2$.*

The *rooted product of G and H on the vertex v* is the graph obtained by identifying each vertex of G with the vertex v of a copy of H [2].

Lemma 2.8 *Let G be a connected graph of order $n \geq 2$, H be a graph with $|V(H)| \geq 2$, and $v \in V(H)$. Then* (i) *if v is not in all maximum independent sets of H, then $\alpha(G(H; v)) = n \cdot \alpha(H)$;* (ii) *if v belongs to every maximum independent set of H, then $\alpha(G(H; v)) = n \cdot (\alpha(H) - 1) + \alpha(G)$.*

Proof. *(i)* Assume $A \in \Omega(H)$ with $v \notin A$, and $S \in \Omega(G(H; v))$. First, $n \cdot \alpha(H) = n \cdot |A| \leq \alpha(G(H; v))$, because the union of n times A is independent in $G(H; v)$. Since S is of maximum size, it follows that, for every copy of H, $S \cap V(H)$ is non-empty and independent. Consequently, we obtain

$$n \cdot \alpha(H) \leq \alpha(G(H; v)) \leq n \cdot \max |S \cap V(H)| \leq n \cdot \alpha(H).$$

(ii) Let $A \in \Omega(G(H; v))$. Then $V(G) \cap A$ is independent in G and

$$\begin{aligned} |A| &= |V(G) \cap A| \cdot \alpha(H) + (n - |V(G) \cap A|) \cdot (\alpha(H) - 1) \\ &= n \cdot (\alpha(H) - 1) + |V(G) \cap A| \end{aligned}$$

On the other hand, one can enlarge a maximum independent set S of G to an independent set U in $G(H; v)$, whose cardinality is

$$\begin{aligned} |U| &= |S| \cdot \alpha(H) + (n - |S|) \cdot (\alpha(H) - 1) \\ &= n \cdot (\alpha(H) - 1) + |S| = n \cdot (\alpha(H) - 1) + \alpha(G). \end{aligned}$$

Since $|V(G) \cap A| \leq \alpha(G)$, we get $\alpha(G(H; v)) = n \cdot (\alpha(H) - 1) + \alpha(G)$.□

By definition, if G is well-covered and $uv \in E(G)$, then u and v belong to different maximum independent sets. Therefore, only isolated vertices, if any, are contained in all maximum independent sets of a well-covered graph. Thus Lemma 2.8*(i)* implies the following.

Corollary 2.9 *If G is a connected graph of order $n \geq 2$, and $H \neq K_1$ is well-covered, then $\alpha(G(H;v)) = n \cdot \alpha(H)$.*

The rooted product of two graphs from $\mathbf{W}_2$ is not necessarily in $\mathbf{W}_2$.

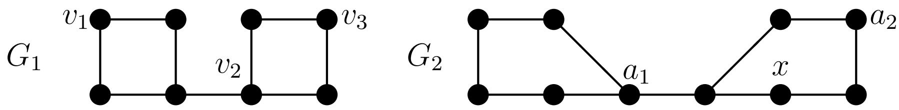

Fig. 2. $G_1 = K_2(C_4;v)$ and $G_2 = K_2(C_5;v)$.

For instance, $K_2, C_5 \in \mathbf{W}_2$, but there is no maximum independent set S in $K_2(C_5;v)$ such that $\{a_1, a_2\} \subset S$ and $x \notin S$, and hence, by Theorem 1.1, the graph $K_2(C_5;v)$ is not in $\mathbf{W}_2$ (see Figure 2). However, $K_2(C_5;v)$ is in $\mathbf{W}_1$, i.e., it is well-covered.

Theorem 2.10 (i) *If $H \in \mathbf{W}_2$, then the graph $G(H;v)$ belongs to $\mathbf{W}_1$.*

(ii) *If $H \in \mathbf{W}_3$, then the graph $G(H;v)$ belongs to $\mathbf{W}_2$.*

Proof. If H is a complete graph, then both *(i)* and *(ii)* are true, according to Propositions 2.1 and 2.2, respectively, because $G(K_p;v) = G \circ K_p$. Assume that H is not complete, and let $V(G) = \{v_i : i = 1, 2, ..., n\}$. By Corollary 2.9, we have $\alpha(G(H;v)) = n \cdot \alpha(H)$.

(i) Let A be a non-maximum independent set in $G(H;v)$. We have to show that A is included in some maximum independent set of $G(H;v)$. Let $S = S_1 \cup S_2 \cup \cdots \cup S_n$, where each S_i is defined as follows:

• S_i is a maximum independent set in the copy H_{v_i} of H;

• $v_i \notin S_i$, whenever $A \cap V(H_{v_i}) = \emptyset$; such S_i exists, since H is well-covered;

• if $v_i \in A \cap V(H_{v_i})$, then $A \cap V(H_{v_i}) \subseteq S_i$; such S_i exists, because H is well-covered;

• if $v_i \notin A \cap V(H_{v_i}) \neq \emptyset$, then $A \cap V(H_{v_i}) \subseteq S_i$ and $v_i \notin S_i$; in accordance with Theorem 1.1, such S_i exists, because H is in $\mathbf{W}_2$. Consequently, $S \in \Omega(G(H;v))$, since all S_i are independent and pairwise disjoint, each one of size $\alpha(H)$, and $A \subset S$. Therefore, $G(H;v)$ is well-covered.

(ii) Let A be a non-maximum independent set in $G(H;v)$ and $x \notin A$. We show that A is included in some maximum independent set of $G(H;v)$ that does not contain the vertex x, and thus, by Theorem 1.1, we obtain that $G(H;v) \in \mathbf{W}_2$. Let $S = S_1 \cup S_2 \cup \cdots \cup S_n$, where S_i is defined as follows:

• S_i is a maximum independent set in the copy H_{v_i} of H;

• if $A \cap V(H_{v_i}) = \emptyset$ and $x \notin V(H_{v_i})$, then $v_i \notin S_i$; such S_i exists, because H is well-covered;

• if $v_i \notin A \cap V(H_{v_i}) \neq \emptyset$ and $x \notin V(H_{v_i})$, then $A \cap V(H_{v_i}) \subseteq S_i$ and $v_i \notin S_i$; such S_i exists, since H is in $\mathbf{W}_2$;

• if $x = v_i$, then $A \cap V(H_{v_i}) \subseteq S_i$ and $v_i \notin S_i$; such S_i exists, because $H \in \mathbf{W}_2$;

• if $x \in V(H_{v_i}) - \{v_i\}$, then $A \cap V(H_{v_i}) \subseteq S_i$ and $x, v_i \notin S_i$; S_i exists, since $A \cap V(H_{v_i}), \{x\}$ and $\{v_i\}$ are independent and disjoint, and H belongs to $\mathbf{W}_3$. Consequently, $S \in \Omega(G(H; v))$ (since all S_i are independent and pairwise disjoint, each one of size $\alpha(H)$), $x \notin S$ and $A \subset S$. Hence, $G(H; v) \in \mathbf{W}_2$. $\square$

3 Conclusions

Lemma 2.5 claims that the function Shed preserves the join operation, i.e., $\mathrm{Shed}(G_1 + G_2) = \mathrm{Shed}(G_1) \cup \mathrm{Shed}(G_2)$. It motivates the following.

Problem 3.1 *Describe graph operations that are preserved by* Shed.

It seems promising to extend our findings in the framework of $\mathbf{W}_k$ classes for $k \geq 3$. Taking into account Theorem 2.10, we propose the following.

Conjecture 3.2 *If $H \in \mathbf{W}_k$, then the rooted product $G(H, v) \in \mathbf{W}_{k-1}$.*

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