Cyclic Automorphism Groups of Graphs and Edge-Colored Graphs *

Mariusz Grech [1] and Andrzej Kisielewicz [2]

Institute of Mathematics
University of Wrocław
Wrocław, Poland

Abstract

In this paper we describe the automorphism groups of graphs and edge-colored graphs that are cyclic as permutation groups. In addition, we show that every such group is the automorphism group of a complete graph whose edges are colored with 3 colors, and we characterize those groups that are automorphism groups of simple graphs.

Keywords: graph, colored graph, automorphism group, cyclic group.

The *König's problem* asks: which abstract groups are isomorphic to the automorphism groups of graphs. It has a simple and easy solution (due to Frucht): every group is abstractly isomorphic to the automorphism group of a suitable graph.

The concrete version of König's problem asks: which permutation groups are the automorphism groups of graphs. This version is very hard. So far it has been solved only for some special classes of the permutation groups.

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[1] Email: `mariusz.grech@math.uni.wroc.pl`

[2] Email: `andrzej.kisielewicz@math.uni.wroc.pl`

The so called *Graphical Regular Representation* problem may be viewed as the concrete König's problem for regular permutation groups. The final result by Godsil [4], obtained in 1979 on the basis of a number of earlier results, provides a full characterization of regular permutation groups that can be represented as the automorphism groups of graphs. In [2], L. Babai has applied the result of Godsil to obtain a similar characterization in the case of directed graphs.

In studying the concrete version of König's problem it turns out that the corresponding results for edge-colored graphs have usually simpler and more natural formulation than their counterparts for simple graphs ([6,7]). This has been noted already in H. Wielandt in [10], where the permutation groups that are automorphism groups of colored graphs and digraphs were called 2^*-closed and 2-closed, respectively. Our results strongly confirm this observation.

1 Introductory results

We study *cyclic permutation groups*, i.e. the groups generated by a single permutation. In [8,9], S. P. Mohanty, M. R. Sridharan, and S. K. Shukla, have considered cyclic permutation groups whose order is a prime or a power of a prime. They gave some partial results. However, in many points the results were wrong or had incorrect proofs. This special case was finally settled by Grech [5], who proved the following.

Theorem 1.1 [5] *Let A be a cyclic permutation group of order p^n, for a prime $p > 2$. Then:*

- *If A has only one nontrivial orbit, then A is not the automorphism group of an edge-colored graph with any number of colors;*
- *If A has exactly two nontrivial orbits, and at least one of them has cardinality $p = 3$ or $p = 5$, then A is the automorphism group of an edge-colored graph with three colors;*
- *Otherwise, A is the automorphism group of a simple graph.*

Theorem 1.2 [5] *Let A be a cyclic permutation group of order 2^n. Then:*

(i) *If A has exactly one orbit of cardinality greater than 2, then A is not the automorphism group of an edge-colored graph with any number of colors;*

(ii) *If A has exactly two nontrivial orbits, one of cardinality 4, and the other of cardinality at least 4, then A is the automorphism group of an edge-colored graph with three colors;*

(iii) *Otherwise, A is the automorphism group of a simple graph.*

As one can see, even in this particular case the problem is non-trivial. Our aim is to extend this result to the cyclic permutation groups of arbitrary order.

2 Preliminaries

We assume that the reader has the basic knowledge in the areas of graphs and permutation groups. Our terminology is standard and follows [1,11].

A *k-colored graph* (or more precisely *k-edge-colored graph*) is a pair $G = (V, E)$, where V is the set of vertices, and E is an *edge-color function* from the set $P_2(V)$ of two elements subsets of V into the set of colors $\{0, \dots, k-1\}$. In other words, G is a complete simple graph with each edge colored by one of k colors. For brevity we write $E(v, w)$ for $E(\{v, w\})$. If $E(v, w) = i$ for some $v, w \in V$ and $i \in \{0, \dots, k-1\}$, then we say that the vertices v and w are *i-neighbors*. The *i-degree* of v is the number of i-neighbors of v. Note that 2-colored graphs can be considered as simple graphs.

An automorphism of a k-colored graph G is a permutation σ of the set V preserving the edge function, that is, satisfying $E(v, w) = E(\sigma(v), \sigma(w))$, for all $v, w \in V$. The group of automorphisms of G will be denoted by $Aut(G)$, and considered as a permutation group acting on the set of the vertices V. By $GR(k)$ we denote the class of automorphism groups of *k-colored graphs*. By GR we denote the union of all classes $GR(k)$ (which is the class of 2^*-closed groups in terms of [10]). Then, $GR(2)$ is the class of automorphism groups of simple graphs.

Permutation groups are treated here up to permutation isomorphism. By C_n we denote the regular action of the cyclic group Z_n. By D_n we mean the group of symmetries of the regular n-gon. It is clear that C_n is a subgroup of D_n (of index two for $n > 2$). Every element of $D_n \setminus C_n$ has order two and is called a *reflection*.

The permutation groups considered here are generated by a single permutation σ, which we write $A = \langle \sigma \rangle$. If $\gamma_1 \dots \gamma_n$ is a decomposition of σ into disjoint cycles, then A has n orbits $O_1, \dots, O_n$, and A restricted to the orbit O_i is the cyclic permutation group.

Let $A = (A, V)$, $B = (B, W)$ be permutation groups acting on disjoint sets V and W. We recall that the *direct sum* $A \oplus B$ of A and B is the permutation group consisting of all pairs (σ, τ), $\sigma \in A, \tau \in B$, acting on $V \cup B$ by the formula $(\sigma, \tau)(x) = \sigma(x)$ for $x \in V$, and $(\sigma, \tau)(x) = \tau(x)$ for $x \in W$.

3 Main result

Our first result characterizes generally the cyclic automorphism groups of edge-colored graphs.

Theorem 3.1 *Let $A = \langle\sigma\rangle$ be a permutation group generated by a permutation σ. Then, A belongs to GR if and only if for every orbit O of σ such that $|O| > 2$, there exists another orbit O' of A such that* $\gcd(|O|, |O'|) > 2$.

The proof of this theorem is by induction on the number of orbits of the permutation group A. Here, to get a flavor of the proofs, we present the first step of the induction.

Lemma 3.2 *Let the permutation group $A = \langle\sigma\rangle$ as above has two orbits O_1 and O_2 such that* $\gcd(|O_1|, |O_2|) > 2$. *Then, $A \in GR$.*

Proof. Let $\gcd(|O_1|, |O_2|) = d > 2$, and $|O_1| = n = dn'$ and $|O_2| = m = dm'$, where n' and m' are coprime. We may assume $O_1 = \{v_0, \ldots, v_{n-1}\}$, $O_2 = \{w_0, \ldots, w_{m-1}\}$ and $\sigma(v_i) = v_{i+1}$, and $\sigma(w_i) = w_{i+1}$, where the indices are taken modulo n and modulo m, respectively.

We will define a graph $H = H(n, m)$ such that $Aut(H) = A$. For $n = m$, this is done in [5, Lemma 3.3]. So, we may assume that $n \neq m$. As the set of vertices we take $V = O_1 \cup O_2$. Then the edge-color function of H is defined as follows.

$$E(e) = \begin{cases} 1 \text{ if } e = \{v_i, v_{i+1}\} \text{ for some } i \in \{0, \ldots, n-1\}, \\ 1 \text{ if } e = \{w_i, w_{i+1}\} \text{ for some } i \in \{0, \ldots, m-1\}, \\ 1 \text{ if } e = \{v_i, w_j\} \text{ and } i \equiv j \pmod{d}, \\ 2 \text{ if } e = \{v_i, w_j\} \text{ and } i \equiv j+1 \pmod{d}, \\ 0 \text{ otherwise.} \end{cases}$$

Thus, the graph H consists of two cycles of color 1 connected by some edges of color 1 or 2 in such a way, that σ (and thus all permutations in A) preserves the colors of the edges. Hence, $A \subseteq Aut(H)$.

Observe that the 1-degree of each vertex in O_1 is equal to $m' + 2$, while the 1-degree of each vertex in O_2 is equal to $n' + 2$. Since $n' \neq m'$, $Aut(H)$ preserves the sets O_1 and O_2. Consequently, since the automorphism group of a simple cycle graph is a dihedral group, $Aut(H) \subseteq D_n \oplus D_m$.

We will show that no permutation (τ, δ), where τ or δ is a reflection, preserves colors of H. It will follow that $Aut(H) \subseteq C_n \oplus C_m$.

By $N_1(v)$, we denote the set of 1-neighbors of the vertex v in the set O_2, and by $N_2(v)$, the set of 2-neighbors of the vertex v in the set O_2. We show that permutations that acts as reflections on some of the sets O_1 or O_2 are forbidden. We take $(\tau, \delta) \in Aut(H))$ that fixes v_0. Observe that $N_2(v_0) = N_1(v_1)$. Moreover, since $n' > 2$, $N_2(v_0) \cap N_1(v_{xy-1}) = \emptyset$. Therefore, (τ, δ) fixes v_1. Since a subgraph of H spanned on O_1 is a $|O_1|$-cycle, (τ, δ) acts trivially on O_1. Since $A \subseteq H$, no reflection on O_1 is possible. Since the role of O_1 and O_2 are symmetric, the same is true for O_2.

To complete the proof, we have to show that if $(\tau, \delta) \in H$ fixes v_0, then (τ, δ) acts as $\sigma_2^{n'l}$ on O_2 for some l. We have $N_1(v_0) = \{w_{n'l}; l \in \{0, \ldots, z-1\}\}$. Therefore, $(\tau, \delta)(w_0) = w_{n'l}$ for some l. Since the subgraph of H spanned on O_2 is a $|O_2|$-cycle, we have that (τ, δ) acts as σ_2^{xl} on O_2. Again the role of O_1 and O_2 are symmetric, therefore, the same is true for O_1 (a permutation that fixes w_0 acts as $\sigma_1^{n'l}$ on O_1 for some l. Thus, $A = Aut(H)$. □

In the full proof of Theorem 3.1, in every construction we use only three colors, therefore we obtain also the following.

Theorem 3.3 *Let A be a cyclic permutation group. Then, $A \in GR(3)$ if and only if $A \in GR$.*

The two above theorems should be compared with the result for cyclic automorphism groups of simple graphs.

Theorem 3.4 *$A \in GR(2)$ if and only if none of the following holds:*

- *There is an orbit O, with $|O| > 2$ such that $\gcd(|O|, |O'|) \leq 2$ for every other orbit O'.*
- *There are two orbits O_1, O_2, with $\gcd(|O_1|, |O_2|) \in \{3, 5\}$ such that $\gcd(|O_1|, |O|) \leq 2, \gcd(|O_2|, |O|) \leq 2$ for every other orbit O.*
- *There are two orbits O_1, O_2, with $\gcd(|O_1|, |O_2|) = 4$ such that $\gcd(|O_1|, |O|) = 1, \gcd(|O_2|, |O|) = 1$ for every other orbit O.*
- *There are three orbits O_1, O_2, O_3, with $\gcd(|O_1|, |O_2|) \in \{3, 5\}, \gcd(|O_3|, |O_2|) \in \{3, 5\}$, $\gcd(|O_1|, |O_3|) \leq 2$ such that $\gcd(|O_1|, |O|) \leq 2, \gcd(|O_2|, |O|) \leq 2, \gcd(|O_3|, |O|) \leq 2$ for every other orbit O.*
- *There are three orbits O_1, O_2, O_3, with $\gcd(|O_1|, |O_2|) = 4$, $\gcd(|O_3|, |O_2|) \in \{3, 5\}$, $\gcd(|O_1|, |O_3) \in \{1, 3, 5\}$ such that $\gcd(|O_1|, |O|) = 1, \gcd(|O_2|, |O|) = 1, \gcd(|O_3|, |O|) = 1$ for every other orbit O.*
- *There are four orbits O_1, O_2, O_3, O_4, with $\gcd(|O_i|, |O_{i+1}) \in \{3, 4, 5\}$, $i \in \{1, 2, 3\}$ and $\gcd(|O_i|, |O_j|) = 1$, otherwise, such that $\gcd(|O_i|, |O|) = 1$ for every other orbit O.*

- *There are four orbits* O_1, O_2, O_3, O_4*, with* $\gcd(|O_1|, |O_2) = 3$*,* $\gcd(|O_1|, |O_3) = 4$*,* $\gcd(|O_1|, |O_4) = 5$ *and* $\gcd(|O_i|, |O_j|) = 1$*, otherwise, such that* $\gcd(|O_i|, |O|) = 1$ *for every other orbit* O*.*

The proof requires several lemmas and will be presented in the full version of the article.

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