Diagonally symmetric polynomials and computational algebra

Conference on Diagonally symmetric polynomials and applications

CIEM, Castro–Urdiales, October 15, 2007

Emmanuel Briand.

Universidad de Sevilla

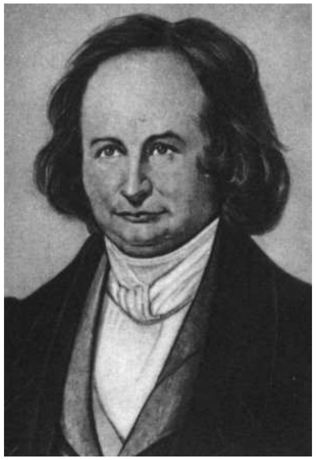
Two topics in computational algebra:

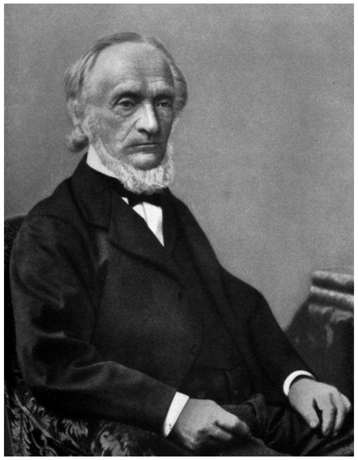
- Solving zero–dimensional systems of multivariate polynomial equations in the projective space $\mathbb{P}(W) = \mathbb{P}(\mathbb{K}^{r+1})$ or affine space $V = \mathbb{K}^r$. We will consider only questions about the *multiset* of the solutions.

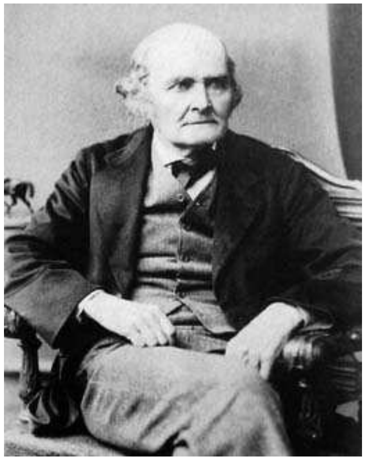
- Determining the forms on W (resp. polynomial functions on V) that *decompose totally*:as product of linear forms (resp. as product of polynomials of degree 1).

These two topics are connected by duality.

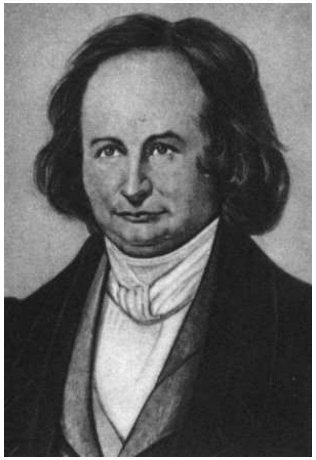
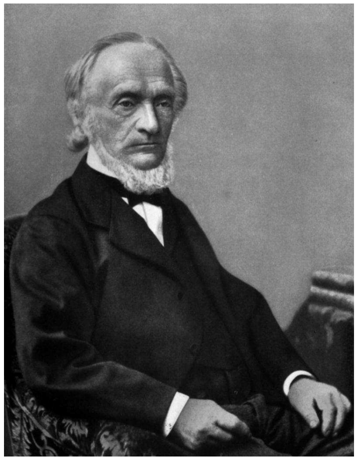
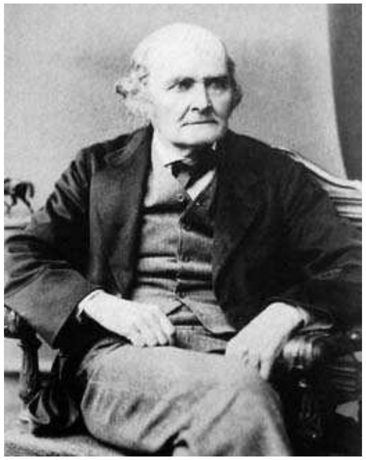
starring







starring


Both problems led first–rank mathematicians (Jacobi, Schläfli, Cayley, Brill) of the XIXth century to compute dsym polynomials.

Homogenous setting

Let $W = \mathbb{K}^{r+1}$ with coordinate functions X_0, X_1, . . . , X_r.

$$\hat{A} = \begin{bmatrix} a_{10} & a_{11} & \cdots & a_{1r} \\ a_{20} & a_{21} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nr} \end{bmatrix}$$

The rows of $\hat{A}$ represent n points in $\mathbb{P}(W)$

$$(a_{i0} : a_{i1} : \cdots : a_{ir})$$

or n hyperplanes in $\mathbb{P}(W^*)$

$$a_{i0}t_0 + a_{i1}t_1 + \cdots + a_{ir}t_r = 0$$

Non–homogeneous setting

In $W = \mathbb{K}^{r+1}$ with coordinate functions X_0, X_1, ..., X_r,

let $\mathbb{A} \sim \mathbb{K}^r$ be the affine space $X_0 = 1$ in $\mathbb{P}(W)$.

$$A = \begin{bmatrix} & a_{11} & \cdots & a_{1r} \\ & a_{21} & \cdots & \vdots \\ & \vdots & & \vdots \\ & a_{n1} & \cdots & a_{nr} \end{bmatrix}$$

The rows of A represent n points in $\mathbb{A}$

$(a_{i1}, a_{i2}, \cdots, a_{ir})$

or n affine hyperplanes of $\mathbb{A}$ (or the corresponding affine functions):

$1 + a_{i1}t_1 + \cdots + a_{ir}t_r = 0$

Diagonally symmetric polynomials (non–homogeneous setting)

$$DSym_n^r(\mathbb{K}) = k[A]^{\mathfrak{S}_n}$$

= polynomials in the entries of A invariants under row permutations.
This is the affine coordinate ring of: *the symmetric product* $\mathbb{A}^n/\mathfrak{S}_n$ which parametrizes:

- multisets of n points of $\mathbb{A}$.
- totally decomposable polynomials $F(t_1, \ldots, t_r)$, of degree $\leq n$ with constant term 1.

Coordinate free–presentation: $T^n_{\mathsf{sym}}\left(\bigoplus_{N=0}^{\infty} S^N V\right)$ where V is the underlying vector space of $\mathbb{A}$.

($T^n_{\mathsf{sym}} M$= symmetric tensors, a subspace of $\otimes^n M$

$S^n M$= symmetric power, a quotient of $\otimes^n M$)

Remarkable families of elements of *DSym*$_n^r$

In *DSym*$_n^r$ there exist

- *monomial functions* (symmetrizations of monomials).

- *power sums* p_α with $\alpha \in \mathbb{N}^r$

- *elementary polynomials* e_α with $\alpha \in \mathbb{N}^r$, $1 \le |\alpha| \le n$. They are defined by their generating function

  $$\sum e_\alpha t^\alpha := \prod_{i=1}^{n} (1 + a_{i1} t_1 + \cdots + a_{ir} t_r)$$

Homogeneous diagonally symmetric polynomials

$\mathit{HDSym}_n^r(\mathbb{K}) \subset k[\hat{A}]^{\mathfrak{S}_n} = \mathit{DSym}_n^{r+1}(\mathbb{K})$

$=$

diagonally symmetric polynomials f with the homogeneity property:

multiplying a row of $\hat{A}$ with λ transforms f into $\lambda^N f$.

This is the homogeneous coordinate ring of *the symmetric product* $\mathbb{P}(W)^n/\mathfrak{S}_n$ which parametrizes:

- multisets of n points of $\mathbb{P}(W)$.

- multisets of n hyperplanes of $\mathbb{P}(W)$ = totally decomposable hypersurfaces of degree n.

Coordinate free–presentation: $\bigoplus_{N=0}^{\infty} T^n_{\mathrm{sym}}\left(S^N W\right)$

Remarkable elements of $HDSym_n^r$

$HDSym_n^r$ is spanned by the monomial functions that are homogeneous.

Its degree 1 component is spanned by the *Fundamental polynomials*

= the homogeneous elementary symmetric polynomials

= the e_α for $\alpha \in \mathbb{N}^{r+1}$ and $|\alpha| = n$.

Their generating series:

$$\sum e_\alpha t^\alpha := \prod_{i=1}^{n} (a_{i0}t_0 + a_{i1}t_1 + \cdots + a_{ir}t_r)$$

Homogeneous vs. nonhomogeneous

$HDSym_n^r \subset DSym_n^{r+1}$

but more relevant is the deshomogenization map:

$$\begin{array}{ccc} HDSym_n^r & \longrightarrow & DSym_n^r \\ a_{i0} & \longmapsto & 1 \end{array}$$

which corresponds to the restriction of functions on $\mathbb{P}(W)^n/\mathfrak{S}_n$ to its affine chart $\mathbb{A}^n/\mathfrak{S}_n$.

It fulfills:

$$e_{\alpha_0,\alpha_1,\cdots,\alpha_r} \longmapsto e \quad {}_{\alpha_1,\ldots,\alpha_r}$$

0–dimensional systems of equations: univariate case

The univariate case: $F(X) = X^n + a_1 X^{n-1} + \cdots + a_n$ Knowing about symmetric functions helps in understanding the roots of $P(X)$ and related objects, *e.g.*:

- Matrix of power sums: $[p_{i+j}]_{i,j=0,\ldots,n-1}$. Assume $\mathbb{K} = \mathbb{C}$.
  - Its rank= # distinct complex root.
  - Its signature= # distinct real roots.
- The global residue

$$\frac{1}{2i\pi} \int \frac{X^{n+k}}{F(X)} dX = h_{k+1}$$

  (complete sum).

0–dimensional systems of equations: univariate case

- *Relations between roots and coefficients*:
  - all symmetric polynomial of the roots are polynomials in the coefficients
  - the coefficients are symmetric polynomials of the roots ($a_k = \pm e_k$).
  - A particular example: Newton identities that relate coefficients and power sums:

$$\frac{XF'(X)}{F(X)} = n + \sum_{k=1}^{\infty} \frac{p_k}{X^k}$$

- Resultants: $Res(P(X), Q(X))$ expresses in several ways in the symmetric functions of the roots of P and symmetric functions of the roots of Q

How much does this generalize in the multivariate setting ?

dsym polynomials of the roots from the coefficients

Two classes of systems where the dsym–polynomial of the roots are polynomial or rational functions of the coefficients:

- *Gröbner bases with prescribed leading terms*: fix a term order $\preceq$ and monomials X^{α_1}, X^{α_2}, . . . , X^{α_k} and consider all systems $f_1 = f_2 = \cdots = f_k = 0$ such that $LT(f_i) = X^{\alpha_i}$.

- *Zero–dimensional complete intersections (ZDCI) without zero at infinity*:

  $f_1 = f_2 = \cdots = f_n = 0$

  (as many equations as unknowns and no common zero for the leading homogeneous parts) in $\mathbb{P}^n$ (or some weighted projective space).

Gröbner bases with prescribed leading term

Gröbner bases with prescribed leading terms: fix a term order $\preceq$ and monomials X^{α_1}, X^{α_2}, . . . , X^{α_k} and consider all systems $f_1 = f_2 = \cdots = f_k = 0$ such that $LT(f_i) = X^{\alpha_i}$.

The generating function of the elementary diagonally symmetric polynomials

$$\sum e_\alpha t^\alpha := \prod_{i=1}^{n} (1 + a_{i1}t_1 + \cdots + a_{ir}t_r)$$

appears as the determinant of

$$\begin{array}{ccc} k[X_1, X_2, \ldots, X_r]/\langle f_1, f_2, \ldots, f_k\rangle & \longrightarrow & k[X_1, X_2, \ldots, X_r]/\langle f_1, f_2, \ldots, f_k\rangle \\ g & \longmapsto & g \cdot (1 + X_1t_1 + \cdots X_rt_r) \end{array}$$

whose matrix in a monomial basis has entries polynomial in the coefficients of the system.

$\Longrightarrow$ the e_α are polynomials in the coefficients of the system.

ZDCI without zero at infinity

There is a “Poisson formula”:

$$\mathrm{Res}(f_1, f_2, \ldots, f_n, f)/\,\mathrm{Res}(h_1, h_2, \ldots, h_n)^N = f(a_1)f(a_2)\cdots f(a_r)$$

where h_i is the leading homogeneous component of f_i.

For $f = 1 + X_1t_1 + \cdots + X_rt_r$, we recover again that the generating function of the elementary functions:

$$\prod_{i=1}^{n}(1 + a_{i1}t_1 + \cdots + a_{ir}t_r) = \sum e_\alpha t^\alpha$$

as

$$\mathrm{Res}(f_1, f_2, \ldots, f_n,\ 1 + X_1t_1 + \cdots + X_rt_r\ )/\mathrm{Res}(h_1, h_2, \ldots, h_n)^N$$

$\Longrightarrow$ the e_α are rational functions in the coefficients of the system.

From elementary polynomials to all dsym–polynomials

In the above situations, the e_α are polynomial or rational functions of the coefficients.

In characteristic big enough ($\mathrm{char}\mathbb{K} > n$ or 0), all dsym–polynomials are polynomials in the e_α, then all dsym–polynomials are polynomial or rational functions of the coefficients.

In small characteristic, the result still holds: use, instead of the generating function of the e_α, the *generating function of the monomial functions*:

$$\sum_{\alpha_1 \preceq \alpha_2 \preceq \cdots \preceq \alpha_n} m[a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_r^{\alpha_n}]\, u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n}$$

$$= \prod_{i=1}^{n} \left(\sum_{\alpha \in \mathbb{N}^n} X^\alpha u_\alpha \right)_{|X = a_i}$$

Intersection of the two classes

A system that is at the same time Gröbner basis and ZDCI with no zero at infinity has the shape (for some $\preceq$):

$$\begin{cases} f_1 = X_1^{d_1} + \cdots \\ f_2 = X_2^{d_2} + \cdots \\ \vdots \\ f_n = X_n^{d_n} + \cdots \end{cases}$$

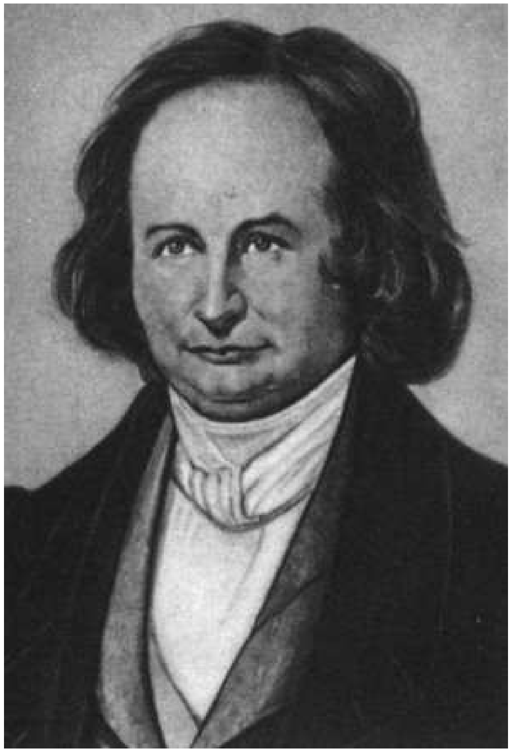
a different approach (multidimensional residues, Tsikh, Aizenberg, Kytmanov) led to a generating series for the *power sums*:

$$p_\alpha = a_1^\alpha + a_2^\alpha + \cdots + a_r^\alpha$$

They are obtained as coefficients in an expansion in Laurent series of:

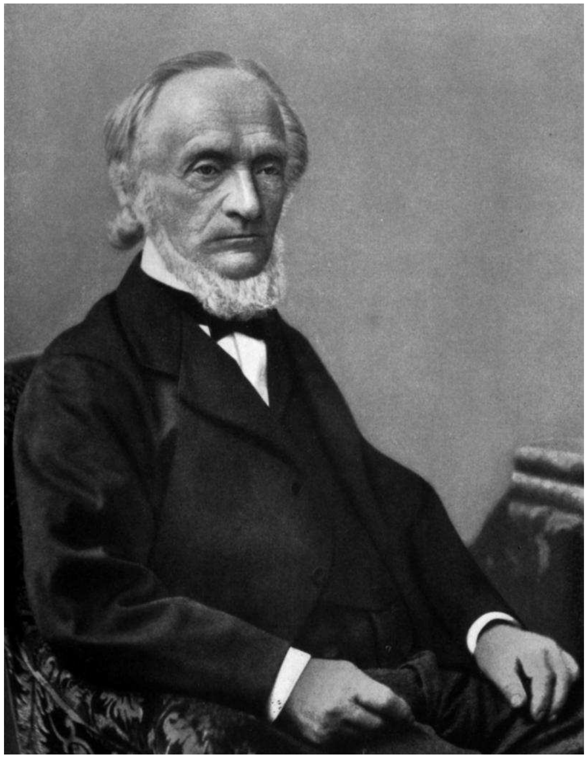
$$\frac{X_1 X_2 \cdots X_n}{f_1 f_2 \cdots f_n} \left| \frac{\partial f}{dx} \right| = \sum_{\alpha \in \mathbb{N}^n} \frac{p_\alpha}{X^\alpha} + \cdots$$

Near formulas were already written in 1835 !



C.G. Jacobi, Theoremata nova algebraica circa systema duarum aequationum inter duas variabiles propositarum, Crelle Journal, 1835

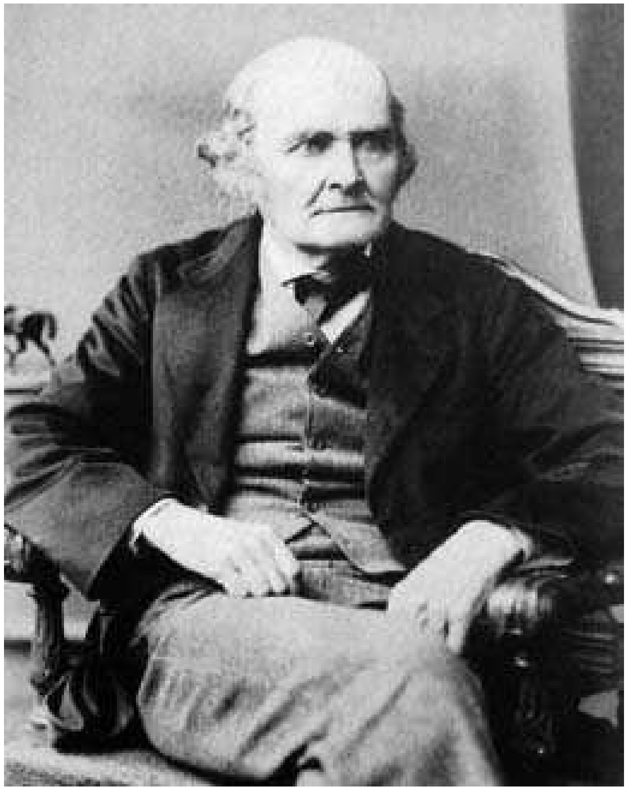
Resultants from dsym polynomials



Schläfli, Über die Resultante eines Systemes mehrerer algebraischen Gleichungen, 1854

Schläfli proposed to use dsym polynomials to express multivariate resultants $Res(F_1, F_2, \ldots, F_r, F_{r+1})$ in the coefficients of the forms F_i: $Res(F_1, F_2, \ldots, F_r, F_{r+1}) = \bullet F_{r+1}(a_1)F_{r+1}(a_2)\cdots F_{r+1}(a_n)$ where the a_i are the roots of $F_1 = F_2 = \cdots = F_r = 0$.

Cayley (1857)



Cayley applied Schläfli's idea for the system

$$\begin{cases} C(X_0, X_1, X_2) = 0 \\ Q(X_0, X_1, X_2) = 0 \\ L(X_0, X_1, X_2) = 0 \end{cases}$$

(cubic, quadratic, linear ternary forms).

Cayley, On the symmetric functions of the roots of certain systems of two equations, Phil. Trans. Royal Soc. London, 1857.

Detailed account of it in *Rota, Stein: A problem of Cayley from 1857 and how he could have solved it, LAA, 2005*.

Cayley's computations

1. express $Res(C, Q, L)$ as a dsym polynomial in the roots of $C = L = 0$.

   $Res(C, L, Q) = \bullet Q(a_1)Q(a_2)Q(a_3)$

   There appear homogeneous monomial functions of degree 2.

2. express the homogeneous monomial functions of degree 2 in term of those of degree 1 (the *Fundamental Polynomials* $e_{\alpha_0,\alpha_1,\alpha_2}$). Cayly needs the hypothesis that a_1, a_2, a_3 are on a same line.

3. Compute the Fundamental functions $e_{\alpha_0,\alpha_1,\alpha_2}$ in the coefficients of C and L. Let E be an indeterminate linear form:

   $E = X_0 t_0 + X_1 t_1 + X_2 t_2$

   then $Res(C, L, E)$ is easily computed:

   $Res(C, L, E) = \bullet Res(L, E, C) = \bullet C(b)$

   where b is the unique solution of $L = E = 0$.

Cayley's question

About step 2: express the homogeneous monomial functions of degree 2 as polynomials in those of degree 1 (the e_α)

What about the monomial functions of higher degrees ?

Rota–Stein's paper: the monomial functions of degrees ≥ 3 are also polynomial functions of the e_α, provided that a_1, a_2, a_3 are on a same line.

Foulkes–Howe conjecture (part of it): in HDSym_n^r, the e_α generate the components of degree $d \geq n$.

Foulkes–Howe conjecture is true for $n = 3$ (E.B.)

Another computational algebra problem: decomposable forms

Setting: ground field $\mathbb{K}$ algebraically closed.

Question 1: Is $F(t_0, t_1, \dots, t_r)$ totally decomposable (= a product of linear forms) ?

Answer: factor F (computer algebraists know how to ask the computer an "absolute factorization").

Question 2: Let $F_u(t_0, \dots, t_r)$ be a form depending on parameters u. For which values of u is F_u totallyt decomposable ?

Raised in *Singer, Ulmer: Linear differential equations and products of linear forms, J. symb. comp., 1997*

Brill's covariant

Brill, Über die Zerfaällung einer Ternärform in Linearfactoren, Math. Ann., 1897

Gordan, Das Zerfallen der Curven in gerade Linien, Math. Ann., 1894

Brill produced a system of equations that defines set–theoretically the subvariety of totally decomposable forms.

Accounts in *Gel'fand, Kapranov, Zelevinsky: Discriminants, resultants and multidimensional determinants, 1994* and *Rota, Stein: a formal theory of resultants, 2001.*

Brill's covariant

Consider the forms $F(t_0, t_1, \ldots, t_r)$ of degree n.

Brill's covariant is a (huge) polynomial $B(F, x)$ depending on the coefficients of F and $3n$ new independant variables with the property that:

$B(F, x) = 0 \Leftrightarrow F$ is totally decomposable

Decompose: $B(F, x) = \sum_\alpha C_\alpha(F) x^\alpha$

Then the system of "Bril's equations":

$C_\alpha(F) = 0 \qquad \forall\alpha$

defines set–theoretically the subvariety of decomposable forms in $S^n W$.

Brill's covariant

Brill's covariant is huge !

Ex: Ulmer–Singer could not use it for quartic forms in 4 variables.

Brill got interested in an alternative solution: dsym polynomials.

Indeed, the equations of the subvariety of totally decomposable forms of degree n in $r+1$ variables

are exactly

the algebraic relations between the Fundamental functions
$e_\alpha \in \mathit{HDSym}_n^{r+1}$

Boring computations . . . ask a graduate student

Brill's student for this task: Friedrich Junker.

He wrote papers of increasing size about dsym polynomials (with many tables of change of basis and many relations between dsym polynomials)

Die Relationen, welche zwischen den elementaren symmetrische Functionen bestehen, Math. Ann., 1890.

Über symmetrische Functionen von mehreren Reihen von Veränderlichen, Math. Ann., 1893.

Die symmetrischen Functionen und die Relationen zwischen der Elementarfunctionen derselben, Math. Ann., 1894.

Die symmetrischen Functionen der semeinschaftlichen variabelnpaaree ternärer Formen, 1897 (50 p. theory + 50 p. tables)

To be continued . . .

Thursday: "Foulkes' Conjecture and Diagonally symmetric polynomials"