Foulkes' conjecture Diagonally symmetric polynomials

Conference on Diagonally symmetric polynomials and applications

CIEM, Castro–Urdiales, October 18, 2007

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Foulkes' plethysm conjecture

Foulkes: concomitants of the quintic and the sextic up to degree four in the coefficients of the ground form, 1950

h_k: complete sums; $f \circ g$: plethysm of symmetric functions.

$$\begin{aligned}
h_3 \circ h_2 - h_2 \circ h_3 &= s_{222} \\
h_4 \circ h_2 - h_2 \circ h_4 &= s_{422} + s_{2222} \\
h_4 \circ h_3 - h_3 \circ h_4 &= s_{732} + s_{6222} + s_{5421} \\
&\vdots
\end{aligned}$$

Foulkes' (open) plethysm conjecture:

For all $N \geq n$ the following holds:
"$h_N \circ h_n - h_n \circ h_N$ is Schur positive" (*FOULKES*(N,n))

Foulkes' plethysm conjecture and representations of $\mathfrak{S}_K$

$\mathfrak{S}_K$, $K = N \times n$.

$h_N \circ h_n$ = Frobenius of $\mathbb{C}P(n^N)$
$h_n \circ h_N$ = Frobenius of $\mathbb{C}P(N^n)$

"$h_N \circ h_n - h_n \circ h_N$ is Schur positive" (*FOULKES*(N, n))
means:

$\mathbb{C}P(N^n)$ can be embedded in $\mathbb{C}P(n^N)$.

characters of the general linear group

$V = \mathbb{C}^d$

$h_N(x_1, \dots, x_d)$ = character of $S^N V$

$=$ trace of *Diag*$(x_1, \dots, x_d)$ in the rep. $S^N V$ of $GL(V)$.

$(h_N \circ h_n - h_n \circ h_N)(x_1, \dots, x_d)$ = character of $S^N(S^n V) - S^n(S^N V)$

An equivariant embedding $S^n(S^N V) \hookrightarrow S^N(S^n V)$

or

equivariant surjection $S^N(S^n V) \to S^n(S^N V)$

(for $V = \mathbb{C}^K$ or for all V) would prove *FOULKES(N,n)*.

Symmetric powers and spaces of symmetric tensors

$S^N V$ = *Symmetric power* of V = a quotient of $\otimes^N V$

Ex: $V = \mathbb{C}^2 = \mathbb{C}x_1 \oplus \mathbb{C}x_2$ then $S^2 V$ has basis x_1^2, $x_1 x_2$, x_2^2 (which reperesent the classes: $[x_1 \otimes x_1]$, $[x_1 \otimes x_2]$, $[x_2 \otimes x_2]$ modulo $x_1 \otimes x_2 \equiv x_2 \otimes x_1$).

$T^N_{\text{sym}} V$ = *symmetric tensors* = a subspace of $\otimes^N V$

Ex: $T^2_{\text{sym}} V$ has basis $x_1 \otimes x_1$, $x_1 \otimes x_2 + x_2 \otimes x_1$, $x_2 \otimes x_2$.

$T^N_{\text{sym}} V \simeq S^N V$ as $GL(V)$–modules.

A graded algebra

$V = W^*$= dual of W.

homogenous polynomial functions of degree N

over $\mathbb{P}(W)$ . . . space $S^N V$.

over $\mathbb{P}(W)^n$ . . . space $\otimes^n S^N V$.

over $\mathbb{P}(W)^n/\mathfrak{S}_n$ . . . space $T^n_{\mathrm{sym}} S^N V$.

Homogeneous coordinate ring of the "symmetric product" $\mathbb{P}(W)^n/\mathfrak{S}_n$:

$$\mathcal{A}_n(V) := \bigoplus_{N=0}^{\infty} T^n_{\mathrm{sym}}\left(S^N V\right)$$

Example in $\mathcal{A}_2(V)$: product of elements $f_1, f_2, g_1, g_2 \in V$, (i.e. of degree 1):

$$(f_1 \otimes f_2 + f_2 \otimes f_1)(g_1 \otimes g_2 + g_2 \otimes g_1) =$$
$$f_1 g_1 \otimes f_2 g_2 + f_1 g_2 \otimes f_2 g_1 + f_2 g_1 \otimes f_1 g_2 + f_2 g_2 \otimes f_1 g_1$$

Howe's ("Foulkes–Howe") conjecture

Howe: (gl_n, gl_m)*–duality and symmetric plethysm, 1987*

$$T^n_{\mathsf{sym}}V \hookrightarrow \mathcal{A}_n(V) = \bigoplus_{N=0}^{\infty} T^n_{\mathsf{sym}}\left(S^N V\right)$$ (piece of degree 1).

Universal property of the symmetric algebra:
$\pi^* : \bigoplus_{N=0}^{\infty} S^N \left(T^n_{\mathsf{sym}}V\right) \longrightarrow \mathcal{A}_n = \bigoplus_{N=0}^{\infty} T^n_{\mathsf{sym}}\left(S^N V\right)$ equivariant map of graded algebras.

It would be nice that the following assertions were true;
FHinj(N,n): "for all V, π^*_N is injective" for all $N \leq n$.
FHsurj(N,n): "for all V, π^*_N is surjective" for all $N \geq n$.

Precisely:
For $n \geq N$: *FHinj*$(N,n) \Rightarrow$ *FOULKES(n,N)*
For $N \geq n$: *FHsurj*$(N,n) \Rightarrow$ *FOULKES(N,n)*

Some remarks

Remember: $\pi^* : \bigoplus_{N=0}^{\infty} S^N\left(T^n_{\text{sym}} V\right) \longrightarrow \mathcal{A}_n = \bigoplus_{N=0}^{\infty} T^n_{\text{sym}}\left(S^N V\right)$

FHinj(N, n): "for all V, π^*_N is injective" *FHsurj*(N, n): "for all V, π^*_N is surjective"

The remarks are:

- *FHinj*$(N, n) \Rightarrow$ *FHinj*$(N-1, n)$ (kernel in degree $N-1$ would imply kernel in degree N).
- *FHinj*$(n, n) =$ *FHsurj*(n, n). (linear map between spaces of the same dimension). "*FH*(n)".

Some results

- Brion (1997): for $N >> n$, *FHsurj*(N, n) is true. (explicit lower bound on N depending on n and r).

- mysterious J.H.: *FH*(3) is true,

- E.B. *FHsurj*$(N, 3)$ is true for all relevant N ($N \geq 3$).

- E.B. (2002), J. Jacob (2004): *FH*(4) is true.

mysterious J.H. proved $FH(3)$ in 1899

Wrong

Müller, Neunhöffer: Some computations regarding Foulkes' conjecture, 2005:

$FH(5)$ is **wrong**.

Is the problem still interesting ?

The geometry behind π^*

Set $V = W^*$.

- The *product map* (n linear forms $\to$ forms of degree n):

  $$\begin{array}{rccc} \pi : & V^n/\mathfrak{S}_n & \longrightarrow & S^n V \\ & f_1, f_2, \ldots, f_n & \longrightarrow & f_1 f_2 \cdots f_n \end{array}$$

  Image of π: subvariety of totally decomposable forms.

- The *union map* (n hyperplanes $\to$ hypersurfaces of degree n):

  $$\mathbb{P}\pi : (\mathbb{P}V)^n/\mathfrak{S}_n \longrightarrow \mathbb{P}(S^n V)$$

  Image of $\mathbb{P}\pi$: subvariety of unions of hyperplanes.

  $\mathbb{P}\pi$ is an embedding (still injective but not an embedding in most modular cases).

$\pi^* : \quad \mathcal{A}_n \longleftarrow \bigoplus_{N=0}^{\infty} S^N T^n_{\mathrm{sym}} V$ is the associated map of graded algebras.

FHinj is still interesting

$$\begin{array}{rrcl}
\pi: & V^n/\mathfrak{S}_n & \longrightarrow & S^nV \\
\mathbb{P}\pi: & (\mathbb{P}V)^n/\mathfrak{S}_n & \longrightarrow & \mathbb{P}(S^nV) \\
\pi^*: & \mathcal{A}_n & \longleftarrow & \bigoplus_{N=0}^{\infty} S^N T^n_{\mathsf{sym}} V
\end{array}$$

ker π^*= ideal of the equations of the variaties of totally decomposable forms and totally decomposable hypersurfaces.

imπ^* = ...see later

FHinj(N,n) = no polynomial of degree N vanishes on the subvariety of totally decomposable forms of degree n.

Ex: *FH(2)* = no polynomial of degree ≤ 2 vanishes on the subvariety of factorizable quadratic forms.

Ex: *FH(3)* = no polynomial of degree ≤ 3 vanishes on the subvariety of products of three linear forms.

mysterious J.H. proved $FH(3)$ in 1899

Jacques Hadamard, Sur les conditions de decomposition des formes, Bulletin de la SMF, 1899

Elementary geometric arguments.

$\mathcal{A}_n(V) = \bigoplus_{N=0}^{\infty} T^n_{\textbf{sym}} S^N V$ in coordinates

Ex: V with basis x_1, x_2, then S^2V has basis: x_1^2, x_1x_2, x_2^2

Then $T^2_{\text{sym}}S^2V$ has basis:
$$\begin{array}{ll} x_1^2 \otimes x_1^2, & x_1^2 \otimes x_2^2 + x_2^2 \otimes x_1^2, \\ x_2^2 \otimes x_2^2, & x_1^2 \otimes x_1x_2 + x_1x_2 \otimes x_1^2, \\ x_1x_2 \otimes x_1x_2, & x_2^2 \otimes x_1x_2 + x_1x_2 \otimes x_2^2. \end{array}$$

Write $x_j(a_i)$ or a_{ij} for x_j in position i, e.g.
$x_1^2 \otimes x_1x_2 = x_1(a_1)^2x_1(a_2)x_2(a_2) = a_{11}^2a_{21}a_{22}$

$$\begin{array}{rrcl} & x_1^2 \otimes x_1x_2 + x_1x_2 \otimes x_1^2 & = & a_{11}^2a_{21}a_{22} + a_{21}^2a_{11}a_{12} \\ & & = & a_1^{20}a_2^{11} + a_2{}^{20}a_1{}^{11} \\ = & m_{(20)(11)} & = & m\left[\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right] \end{array}$$

Homogeneous monomial function of degree 2: $|(20)| = |(11)| = 2$.

$$\begin{array}{rcl} m_{(20)(01)} & = & a_1^{20}a_2^{01} + a_2^{20}a_1^{01} \\ \left(m_{(20)(01)}\right)_{|a_1=\lambda a_1} & = & \lambda^2 a_1^{20}a_2^{01} + \lambda a_2^{20}a_1^{01}. \end{array}$$

Not homogeneous.

π and π^* in coordinates

The homogeneous monomial functions of degree 1 are the "Fundamental" (= homogeneous elementary) functions: $e_{\alpha_1,\ldots,\alpha_d}$ with $|\alpha| = n$.

Ex: $n = 2$, $\dim V = 2$,
In $\mathcal{A}_2$:
$x_1 \otimes x_1 = e_{20}$ (2 ocurrences of x_1, 0 of x_2)
$x_2 \otimes x_2 = e_{02}$
$x_1 \otimes x_2 + x_2 \otimes x_1 = e_{11}$

In $\displaystyle\bigoplus_{N=0}^{\infty} S^N T^n_{\text{sym}} V$ (algebra of polynomials), the same objects should be considered as independant variables: $x_1 \otimes x_1 = Y_{20}$
$x_2 \otimes x_2 = Y_{02}$
$x_1 \otimes x_2 + x_2 \otimes x_1 = Y_{11}$

Then $\pi^* : \quad Y_\alpha \mapsto e_\alpha$.

ker π^* and *im* π^* in coordinates

ker π^* $=$ algebraic relations between the Fundamental functions e_α.

im π^* $=$ e*–decomposable* dsym polynomials $=$ polynomials in the Fundamental functions e_α.

FHinj(N, n) $=$ no relation up to degree N.

FHsurj(N, n) $=$ all homogeneous dsym $(\S_n)$ polynomials of degree N are "e–decomposable".

For which $N \geq n$ are *FHsurj*(N, n) true ? (see Schläfli+Cayley's approach to compute resultants).

Junker's Tools: (i) Polarization

Remember: the variables are $a_{ij} = x_j(a_i)$ and $\mathfrak{S}_n$ permutes the a_i.

The $\mathfrak{S}_n$–invariant polarization from k to j:

$P[jk] = \sum_{i=1}^{n} x_j \frac{\partial}{dx_k}$ transforms dsym polynomials into dsym polynomials.

Ex:

$$\begin{aligned} P[21]\,(a_{11}a_{12}) &= P[11]\,(x_1(a_1)x_1(a_2)) \\ &= x_2(a_1)x_1(a_2) + x_1(a_1)x_2(a_2) \\ &= a_{12}a_{21} + a_{11}a_{22} \end{aligned}$$

i.e. $P[21]e_2(a_1, a_2) = e_{11}(a_1, a_2)$.

Ex: the identity between symmetric polynomials $p_2 = e_1^2 - 2\,e_2$ becomes, after applying $P[21]$:

$2\,p_{11} = 2\,e_{10}\,e_{01} - 2\,e_{11}$

Junker's Tools: (ii) Contraction

Remember: the variables are $a_{ij} = x_j(a_i)$, $\S_n$ permutes the a_i.

Contraction $C[kj]$ = Replace x_j with x_k.

More general tool: replace x_j with $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$

Ex: the decomposition of the permanent:

$$\begin{aligned} e_{111} \;=\; & p_{111} - p_{110}\, p_{001} - p_{101}\, p_{010} - p_{011}\, p_{100} \\ & + 2\, p_{100}\, p_{010}\, p_{001} \end{aligned}$$

provides, after evaluation at $x_1 = x_1^4$, $x_2 = x_1^2$, $x_3 = x_1$, the decomposition of the (ordinary) monomial function:

$$\begin{aligned} m_{421} & = p_{4+2+1} - p_{4+2}\, p_1 - p_{4+1}\, p_2 - p_{2+1}\, p_4 + 2\, p_4\, p_2\, p_1 \\ & = p_7 - p_6\, p_1 - p_5\, p_2 - p_4\, p_3 + 2\, p_4\, p_2\, p_1 \end{aligned}$$

Checking *FHsurj*(N,n): first reduction

FHsurj(N,n) $=$ all monomial functions of degree N in *HDSym*$_n^d$ (all d) are e–decomposable.

- Any such monomial function is obtained by contraction from a multilinear homogeneous monomial function $m_{\star(N,n)}$.

  Ex: $m_{\star(3,2)} = m \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ Ex: $m_{\star(2,3)} = m \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- Contraction sends fundamental functions to fundamental functions. Ex: $C[12]e_{11} = 2\,e_2$.

Thus: $\boxed{m_{\star(N,n)} \quad e\text{–decomposable} \quad \Rightarrow \quad \textit{FHsurj}(N,n).}$

Ex: $m \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(e_{1100}e_{0011} - e_{1010}e_{0101} + e_{1001}e_{0110})$ is a certificate for $FH(2)$.

Checking *FHsurj*(N,n): the multilinear certificate

$$\boxed{m_{\star(N,n)} \quad e\text{–decomposable} \quad \Rightarrow \quad \textit{FHsurj}(N,n).}$$

Ex: $m\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\left(e_{1100}e_{0011} - e_{1010}e_{0101} + e_{1001}e_{0110}\right)$ is a certificate for $FH(2)$.

Apply $C[13]$ and $C[24]$:

$$m\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = e_{11}^2 - 2\,e_{20}e_{02}$$

(which is just an homogenization of $p_2 = e_1^2 - 2\,e_2$)

Decomposing $m_{\star(N,n)}$

$m_{\star(N,n)} = \sum ? e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_N}$

By brute force: write the matrix $M(N,n)$ whose columns give the decompositions of the multilinear products of e_α in the monomial basis. It is a matrix of the restriction of π^* (for $V = \mathbb{C}^{Nn}$) to the multilinear pieces.

$M(N,n)$ full rank *iff* *FHsurj*(N,n) holds. (assume $N \geq n$)

Ex: $N = n = 2$, the matrix is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

In the multilinear piece of *HDSym*$^2_4$,

Products of Fundamental functions: $e_{1100}e_{0011}$, $e_{1010}e_{0101}$, $e_{1001}e_{0110}$.

Monomial functions: $m\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$, $m\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $m\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The matrix $M(N,n)$

In the multilinear piece of *HDSym*$_n^{Nn}$,

Products of Fundamental functions (ex: $N=n=2$, they are $e_{1100}e_{0011}$, $e_{1010}e_{0101}$, $e_{1001}e_{0110}$)	$\leftrightarrow$	$P(n^N)$ (set partitions in N blocks of size n)
Monomial functions (ex: $N=n=2$, they are $m\begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix}$, $m\begin{bmatrix}1&0\\0&1\\1&0\\0&1\end{bmatrix}$, $m\begin{bmatrix}1&0\\0&1\\0&1\\1&0\end{bmatrix}$.)	$\leftrightarrow$	$P(N^n)$ (set partitions in n blocks of size N)

Entries of the matrix: coeff of m_P in $e_Q = 1$ if $P \wedge Q = \hat{0}$, 0 else.

$\boxed{\text{This is the Matrix of Black and List.}}$

The second certificate

- *FH*$(N, n) \quad \Leftrightarrow \quad m_{\star(N,n)}$ is e–decomposable.
- The monomial function $m_{\star(N,n)} \in$ *HDSym*$_n^{Nn}$ is obtained by polarisation from the monomial function $m_{(N\varepsilon_1)(N\varepsilon_2)\cdots(N\varepsilon_n)}$.
- polarisation preserves e–decomposability.

Ex: $N = n = 2$.

$$m_{(2\varepsilon_1)(2\varepsilon_2)} = m \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = e_{11}^2 - 2\, e_{20} e_{02}$$

Apply: $P[31]P[42]$, it yields the multilinear certificate:

$$4\, m_{\star(2,2)} = 4\, m \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = 2\, e_{0110} e_{1001} + 2\, e_{1100} e_{0011} - 2\, e_{2000} e_{0101}$$

$$\boxed{\textit{FH}(N, n) \quad \Leftrightarrow \quad m_{(N\varepsilon_1)(N\varepsilon_2)\cdots(N\varepsilon_n)} \text{ is } e\text{–decomposable.}}$$

A second symmetric group

In *HDSym*$_n^N$:

Remember the variables are $a_{ij} = x_j(a_i)$. The group $\mathfrak{S}_n$ permutes the a_i, the dsym polynomials are its invariants. A second group $\mathfrak{S}_N$ permutes the x_j, it acts on *HDSym*$_n^N$ and there permutes the e_α.

Ex: $\tau_{12} \cdot e_{3,6,9} = e_{6,3,9}$.

The monomial function $m_{(N\varepsilon_1)(N\varepsilon_2)\cdots(N\varepsilon_n)}$ is invariant. If it decomposes in fundamental functions, it should also admit a symmetric decomposition (average !).

$m_{(N\varepsilon_1)(N\varepsilon_2)\cdots(N\varepsilon_n)}$ is e–decomposable *iff* it is a linear combination of symm

Doubly symmetric polynomials of degree N

Linear basis: orbit sums of monomial functions:

Ex ($n = N = 3$):

$$\widetilde{m}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = m\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} + m\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} + m\begin{bmatrix} 0 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

The functions M are indexed with classes of $n \times n$ matrices with row sums N and column sums N, modulo $\mathfrak{S}_n \times \mathfrak{S}_N$.

Orbit sums of functions e:

Ex ($N = n = 3$):

$$E\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = e_{300}e_{012}e_{021} + e_{030}e_{201}e_{102} + e_{003}e_{210}e_{120}$$

The functions E are indexed with classes of $n \times N$ matrices with row sums N and columns sums n, modulo $\mathfrak{S}_n \times \mathfrak{S}_N$.

Doubly symmetric certificate

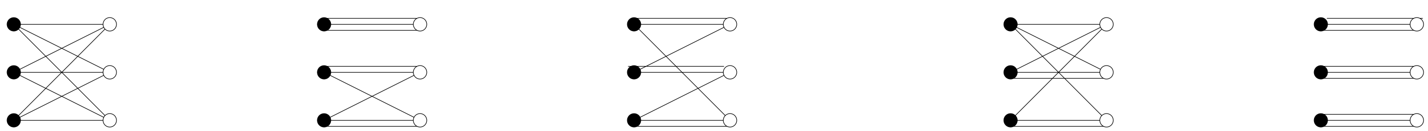
FHsurj(N,n) $\Leftrightarrow$ $m_{(N\varepsilon_1)(N\varepsilon_2)\cdots(N\varepsilon_n)}$ is linear combination of functions E

Ex: $FH(2)$

$$m_{(2\varepsilon_1)(2\varepsilon_2)}e_{11}^2 - 2\,e_{20}e_{02} = E\begin{bmatrix}1&1\\1&1\end{bmatrix} - 2\,E\begin{bmatrix}2&0\\0&2\end{bmatrix}.$$

Ex: $FH(3)$:

$$m\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix} =$$
$$E\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} - 3\,E\begin{bmatrix}3&0&0\\0&2&1\\0&1&2\end{bmatrix} + 6\,E\begin{bmatrix}2&1&0\\0&2&1\\1&0&2\end{bmatrix} - 3\,E\begin{bmatrix}1&1&1\\1&2&0\\1&0&2\end{bmatrix} + 33\,E\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}.$$



Obtaining the doubly symmetric certificate

In the degree N piece of $\mathbb{C}[\{x_j(a_i)\,|\,i=1,\ldots,n;j=1,\ldots,N\}]^{\mathfrak{S}_n\times\mathfrak{S}_N}$, consider the matrix $T(N,n)$ whose columns represent the decomposition of the functions E in the symmetrizations M of monomial functions.

Ex:

$$T(2,2)=\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}$$

$$T(3,3)=\begin{bmatrix}0&0&0&0&1\\0&0&0&1&3\\0&0&1&3&3\\0&1&2&5&6\\1&9&6&18&12\end{bmatrix}$$

$T(4,4)$ has order 43, is invertible but not triangular.

Müller+Neunhoffer: $T(5,5)$ is not invertible.

A theorem

FHsurj(N,3) is true for all $N \geq 3$.

cf. Cayley–Stein problem.

Proof: by induction on N.

FHsurj$(N,3) \Rightarrow$ ***FHsurj***$(N+1,3)$

All monomial function of degree N is e–decomposable $\Rightarrow$
$m_{(3\varepsilon_1)(3\varepsilon_2)(3\varepsilon_3)}$ is linear combination of functions E.

$$M\begin{bmatrix} N+1 & 0 & 0 \\ 0 & N+1 & 0 \\ 0 & 0 & N+1 \end{bmatrix} = e_{1,1,1}\, M\begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix} - M\begin{bmatrix} 0 & N & 1 \\ 0 & 1 & N \\ N+1 & 0 & 0 \end{bmatrix} - M\begin{bmatrix} N & 1 & 0 \\ 0 & N & 1 \\ 1 & 0 & N \end{bmatrix}.$$

$$\widetilde{m}\begin{bmatrix} 0 & N & 1 \\ 0 & 1 & N \\ N+1 & 0 & 0 \end{bmatrix} =$$
$$e_{2,0,1}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 0 \\ 1 & N & 0 \\ 0 & 0 & N \end{bmatrix} - e_{1,0,2}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 1 \\ 1 & N & 0 \\ 0 & 0 & N-1 \end{bmatrix} + e_{0,0,3}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 2 \\ 1 & N & 0 \\ 0 & 0 & N-2 \end{bmatrix}.$$

$$3\,\widetilde{m}\begin{bmatrix} N & 1 & 0 \\ 0 & N & 1 \\ 1 & 0 & N \end{bmatrix} = e_{2,1,0}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 0 \\ 0 & N & 0 \\ 1 & 0 & N \end{bmatrix} + 2\,e_{2,0,1}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 0 \\ 0 & N & 1 \\ 1 & 0 & N-1 \end{bmatrix} -$$
$$e_{1,1,1}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 1 \\ 0 & N & 0 \\ 1 & 0 & N-1 \end{bmatrix} - e_{1,0,2}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 1 \\ 0 & N & 1 \\ 1 & 0 & N-2 \end{bmatrix} + e_{0,1,2}\,\widetilde{m}\begin{bmatrix} N-1 & 0 & 2 \\ 0 & N & 0 \\ 1 & 0 & N-2 \end{bmatrix}.$$

End