Permutation representations on $(0, 1)$ invertible matrices

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Abstract

We present two families of permutations representations. One of them is a generalization of the conjugacy representation of S_n while the other is an interpolation between natural representations of $S_n \times S_n$. We compute characters and present combinatorial formulas of multiplicities of irreducible representations in our representations.

The regular representation

A group G acts on itself by left multiplication:

$x^g = gx$.

The conjugacy representation:

A group G acts on itself by conjugacy:

$x^g = gxg^{-1}$.

Fact: Every irreducible representation ρ of G appears in the regular representation $dim\rho$ times.

Theorem. (Frumkin, 1986): Every irreducible representation of S_n appears in the conjugacy representation at least once.

Two permutations on $GL(n, \mathbb{F})$

Action α:

$G = S_n \times S_n$, $\mathbb{F} =$ any field.

G acts on $GL_n(\mathbb{F})$ by:

$$(\pi, \sigma) \bullet A = \pi A \sigma^{-1}$$

Action β:

$G = S_n = \{(\pi, \pi) \mid \pi \in S_n\} \subset S_n \times S_n$

G acts on $GL_n(\mathbb{F})$ by:

$$(\pi, \pi) \circ A = \pi A \pi^{-1}$$

For $M \subset GL_n(\mathbb{F})$ closed under the action α:
$\alpha_M =$ permutation representation of $S_n \times S_n$ on M.

$\beta_M =$ permutation representation of S_n on M.

Examples of subsets closed under α

$o(A)$ = number of nonzero entries in A.

$\eta(A) \vdash o(A)$ = row sum vector.

$\theta(A) \vdash o(A)$ = column sum vector.

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\eta(A) = (4, 3, 1, 1) \vdash 9$$

$$\theta(A) = (3, 3, 2, 1) \vdash 9.$$

Examples of subsets closed under α (Cotd.)

$U_{n,k}$ is the $n \times n$ matrix :

Upper left $k \times k$ block: upper triangular with the upper triangle filled by ones.

Upper right $k \times (n-k)$ block is filled by ones.

Lower left $(n-k) \times k$ block: zero matrix.

Lower right $(n-k) \times (n-k)$ block: identity matrix I_{n-k}.

Example:

$$U_{7,3} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark The matrix $U_{n,n-1} = U_{n,n}$ is the upper triangular matrix whose upper triangle is filled by ones.

Example:

$$U_{7,7} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Examples of subsets closed under α (Cotd.)

Define

$$H_n^k = \{\pi U_{n,k}\sigma \mid \pi, \sigma \in S_n\}$$

For $A \in H_n^k$:

$$\eta(A) = (n, n-1, \ldots, n-(k-1), 1^{n-k})$$

$$\theta(A) = ((k+1)^{n-(k-1)}, k, k-1, \ldots, 2, 1).$$

Easy to proof that H_n^k consists of exactly those matrices A whose $\eta(A)$ and $\theta(A)$ are as above.

Remark

$$|H_n^k| = n!(n)_k$$

where $(n)_k = \binom{n}{k}k!$.

Remark $H_n^{n-1} = H_n^n$

$$\begin{aligned} H_n^n &= \{\pi U_{n,n}\sigma \mid \pi, \sigma \in S_n\} \\ &= \{A \in GL_n(\mathbb{Z}_2) \mid \\ &\quad \eta(A) = \theta(A) = (n, n-1, n-2, \ldots, 2, 1)\}. \end{aligned}$$

$$|H_n^n| = (n!)^2 = |S_n \times S_n|$$

Theorem: The representation $\alpha_{H_n^n}$ is isomorphic to the regular representation of $S_n \times S_n$.

Proof: Define a bijection $H_n^n \longleftrightarrow S_n \times S_n$ by:

$$\pi U_{n,n} \sigma \mapsto (\pi, \sigma^{-1}).$$

Since each row (column) of $U_{n,n}$ has a different number of 1-s (from 1 to n), we have:
$\pi_1 U_{n,n} \sigma_1 = \pi_2 U_{n,n} \sigma_2 \iff \pi_1 = \pi_2$ and $\sigma_1 = \sigma_2$.
So, φ is well-defined and bijective. Now:

$$(\omega, \tau) \bullet (\pi U_{n,n} \sigma) =$$
$$= \omega \pi U_{n,n} \sigma \tau^{-1} \overset{\varphi}{\mapsto} (\omega \pi, \tau \sigma^{-1})$$
$$= (\omega, \tau)(\pi, \sigma^{-1}).$$

Thus φ is an isomorphism of $S_n \times S_n$ - modules between H_n^n and the (left) regular representation of $S_n \times S_n$. $\square$

A mapping from H_n^k to S_n:

Define $t : H_n^k \longrightarrow S_k$ by

$$\pi U_{n,k}\sigma \mapsto \pi\sigma.$$

t is a surjection preserving the action α of $S_n \times S_n$ and the action β of S_n. Thus, t gives rise to an epimorphism between $\beta_{H_n^k}$ and the conjugacy representation of S_n.

Theorem:

Every irreducible representation of S_n is a constituent in $\beta_{H_n^k}$.

Characters of α and β:

For every $M \subset GL_n(\mathbb{F})$ closed under the action α:

$$\begin{aligned}\chi_{\alpha_M}(\pi, \sigma) &= \#\{A \in M \mid \pi A \sigma^{-1} = A\} \\ &= \#\{A \in M \mid \pi = A \sigma A^{-1}\}\end{aligned}$$

Fact.

Two permutations π, σ are conjugate iff they are similar as matrices.

Corollary.

For every finite set $M \subseteq GL_n(\mathbb{F})$ invariant under the action α of $S_n \times S_n$:

If π and σ are conjugate in S_n, then

$$\begin{aligned}\chi_{\alpha_M}((\pi, \sigma)) = \chi_{\alpha_M}((\pi, \pi)) = \\ = \chi_{\beta_M}(\pi) = \#\{A \in M \mid \pi A = A \pi\}\,.\end{aligned}$$

If π is not conjugate to σ in S_n, then

$$\chi_{\alpha_M}((\pi, \sigma)) = 0\,.$$

Example:

Take $M = S_n$ (embedded in $GL_n(\mathbb{F})$ as permutation matrices). In this case β_M is just the **conjugacy representation of S_n** and a direct calculation shows that for every $\pi \in S_n$:

$$\chi_{\beta_M}(\pi) = |C_\pi| = \frac{n!}{|C(\pi)|} = \chi_{\alpha_M}(\pi, \pi).$$

For every irreducible representation of S_n corresponding to a partition $\lambda \vdash n$ one has:

$$\begin{aligned} m(\lambda, \beta_M) &= \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi) \chi_{\beta_M}(\pi) \\ &= \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi) \frac{n!}{|C(\pi)|} \\ &= \sum_{C \in \hat{S}_n} |C| \chi_\lambda(C) \frac{1}{|C|} = \sum_{C \in \hat{S}_n} \chi_\lambda(C) \; . \end{aligned}$$

Also by direct calculation,

$$m\big((\lambda,\lambda),\alpha_M\big) = 1,$$

and

$$m\big((\lambda,\mu),\alpha_M\big) = 0 \quad \text{when } \lambda \neq \mu.$$

This means that $\alpha_M \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes S^\lambda$ where S^λ is the irreducible S_n- module corresponding to λ.

Characters of α and β on H_n^k.

Theorem:

$$\chi_{\beta_{H_n^k}}(\pi) = \chi_{\alpha_{H_n^k}}(\pi,\pi) = |C_\pi|(n - |supp(\pi)|)_k$$
$$= \chi_{Conj}(\pi)(n - |supp(\pi)|)_k.$$

where

$$\chi_{Conj}(\pi) = |C_\pi| = \text{conjugacy character of } S_n$$

and

$$supp(\pi) = \{i \in [1..n] \mid \pi(i) \neq i\}.$$

Proof: Denote $t = |supp(\pi)|$. Since the character is a class function, we can assume that π is of the following form:

$$\pi = \begin{pmatrix} \pi_t & 0 \\ 0 & I_{n-t} \end{pmatrix},$$

where $\pi_t \in S_t$ has no fixed points.

We have to calculate the number of matrices $A \in H_n^k$ which commute with π. Recall that

$$H_n^k = \{A \,|\, \eta(A) = (n, n-1, \ldots, n-(k-1), 1^{n-k}),$$
$$\theta(A) = ((k+1)^{n-k}\,,\, k\,,\, (k-1)\,, \ldots,\, 2\,,\, 1)\}$$

For every $A \in H_n^k$, denote by $\delta(A)$ the row sums vector of A and by $\varepsilon(A)$ the column sums vector of A.

For example:

If

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in H_4^2$$

then $\delta(A) = (1, 3, 1, 4) \vDash 9$ and
$\varepsilon(A) = (3, 2, 3, 1) \vDash 9$.

Note that every $A \in H_n^k$ has k rows with row sums ranging from n to $n-k+1$, these will be called 'long rows'.

The other $n-k$ rows are monomial.

Likely, A has k columns with column sums ranging from 1 to k, these will be called 'short columns'.

The other $n-k$ columns have $k+1$ ones each.

In πA only the first t rows of A are permuted while in $A\pi$ only the first t columns of A arepermuted.

Note also that for every $\pi \in S_n$ we have: $\delta(A\pi) = \delta(A)$ and $\varepsilon(\pi A) = \varepsilon(A)$.

Since all the ‘long’ rows of A have different row sums, if one of the first t rows of A is ‘long’ then

$$\delta(\pi A) \neq \delta(A) = \delta(A\pi)$$

and thus $A\pi \neq \pi A$.

Hence, we can assume that all the 'long' rows in A are located after the first t rows of A.

This implies that the first t rows of A are monomial.

By similar arguments, the 'short' columns are located after the first t columns.

The upper right $t \times (n-t)$ block of A is the zero matrix.

Indeed, if $A_{i,j} = 1$ for some $1 \le i \le t$ and $t+1 \le j \le n$ then for each $1 \le i' \le t$ with $i' \ne i$ we have:
$A_{i',j} \ne 1$ since A is invertible.

Now, in πA this 1 moves to another place while in $A\pi$ it is left in its original position.

We have now that the upper left $t \times t$ block of A is a permutation matrix which commutes with π_t in S_t .

By similar arguments, each row of the lower left $(n-t) \times t$ block must be or completely filled by 1's or completely filled by zeros while the lower right $(n-t) \times (n-t)$ block is a matrix from H_{n-t}^k.

C_π^t denotes the centralizer subgroup of the element π in S_t

Now calculate:

$$\begin{aligned}
\chi_{\beta_{H_n^k}}(\pi) &= \#\{A \in H_n^k \mid \pi A = A\pi\} \\
&= |C_\pi^t||H_{n-t}^k| = |C_\pi^t|(n-t)!(n-t)_k \\
&= |C_\pi|(n-t)_k = (n - |\mathrm{supp}(\pi)|)_k \chi_{\mathrm{Conj}}(\pi)\,.
\end{aligned}$$

The decomposition of β into irreducibles

Theorem:

Every irreducible representation of S_n is a constituent in $\beta_{H_n^k}$.

Proposition:

Let $\lambda \vdash n$.

$$m\big(\lambda, \beta_{H_n^k}\big) = \sum_{C \in \hat{S}_n} \chi_\lambda(C)(n - |supp(C)|)_k \ ,$$

where $\hat{S}_n$ denotes the set of conjugacy classes of S_n.

Asymptotic behavior of $\beta_{H_n^k}$

Theorem for conj. repr. of S_n (Roichman, 97): $m(\lambda)$ = multiplicity of the irreducible representation S^λ in the conjugacy representation of S_n. f^λ = degree of S^λ.
Then for any $0 < \varepsilon < 1$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} \leq \delta(\varepsilon)$,

$$1 - \varepsilon < \frac{m(\lambda)}{f^\lambda} < 1 + \varepsilon.$$

Theorem for our repr. $\beta_{H_n^k}$: Under the conditions of the above Roichman's theorem, for any $k \leq n$

$$1 - \varepsilon < \frac{m(\lambda, \beta_{H_n^k})}{(n)_k f^\lambda} < 1 + \varepsilon.$$

Proof: In Roichman's work it is shown that under the above conditions

$$|m(\lambda) - f^\lambda| = \left| \sum_{C \in \hat{S}_n} \chi_\lambda(C) - f^\lambda \right|$$
$$= \left| \sum_{C \neq id} \chi_\lambda(C) \right| \leq \varepsilon f^\lambda,$$

which immediately implies the above Roichman's Theorem.

In our case we have the trivial observation $(n - |\text{supp}(C)|)_k \leq (n)_k$ which together with the above gives us:

$$|m(\lambda, \beta_{H_n^k}) - (n)_k f^\lambda| =$$

$$= \left| \sum_{C \in \hat{S}_n} \chi_\lambda(C)(n - |\text{supp}(C)|)_k - (n)_k f^\lambda \right|$$

$$= \left| \sum_{C \neq id} \chi_\lambda(C)(n - |\text{supp}(C)|)_k \right| \leqslant$$

$$\leqslant (n)_k \left| \sum_{C \neq id} \chi_\lambda(C) \right| \leq (n)_k \varepsilon f^\lambda,$$

and our claim is proved.

Theorem for conj. character of S_n (Adin, Frumkin, 86)

$\chi_R^{(n)}$=regular character of S_n.
$\chi_{Conj}^{(n)}$ = conjugacy character of S_n.

$$\lim_{n\to\infty} \frac{\|\chi_R^{(n)}\|}{\|\chi_{Conj}^{(n)}\|} = \lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{Conj}^{(n)} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{Conj}^{(n)}\|} = 1$$

where $\| \quad \|$ denotes the norm with respect to the standard scalar product of characters.

Theorem for character of $\beta_{H_n^k}$

$$\lim_{n\to\infty} \frac{\|(n)_k \chi_R^{(n)}\|}{\|\chi_{\beta_{H_n^k}}\|} = 1\,,$$

$$\lim_{n\to\infty} \frac{\langle (n)_k \chi_R^{(n)}, \chi_{\beta_{H_n^k}} \rangle}{\|(n)_k \chi_R^{(n)}\| \cdot \|\chi_{\beta_{H_n^k}}\|} = \lim_{n\to\infty} \frac{\langle \chi_R^{(n)}, \chi_{\beta_{H_n^k}} \rangle}{\|\chi_R^{(n)}\| \cdot \|\chi_{\beta_{H_n^k}}\|} = 1\,,$$

where k is bounded or tends to infinity remaining less than n.

The representations α_M for $M = H_n^k$

$\alpha_{H_n^0} \cong \bigoplus_{\lambda \vdash n} S^\lambda \otimes S^\lambda$,

$\alpha_{H_n^n} \cong \bigoplus_{\lambda,\rho \vdash n} f^\lambda f^\rho S^\lambda \otimes S^\rho$.

$\alpha_{H_n^k}$ can be seen as a type of an interpolation between these two representations.

Proposition:

For any n and any $0 \le k \le n$,

$$m\left((\lambda,\mu), \alpha_{H_n^k}\right) = \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi)\chi_\mu(\pi)(n - |supp(\pi)|)_k.$$

A combinatorial view of $\alpha_{H_n^k}$

Definition: Define the following subset of H_n^k:

$$W_n^k = \{\pi_k \pi_{n-k} U_{n,k} \sigma_k \sigma_{n-k} \mid \pi_k, \sigma_k \in S_k \text{ and } \pi_{n-k}, \sigma_{n-k} \in S_{n-k}\}.$$

W_n^k = orbit of the matrix $U_{n,k}$ under the action α, restricted to the subgroup $(S_k \times S_{n-k}) \times (S_k \times S_{n-k})$.

$\omega_{n,k}$ = permutation representation of $(S_k \times S_{n-k}) \times (S_k \times S_{n-k})$ on W_n^k, corresponding to the action α .

Proposition:

$$\omega_{n,k} \cong R_k \otimes \left(\bigoplus_{\rho \vdash n-k} S^\rho \otimes S^\rho \right),$$

where R_k is the regular representation of $S_k \times S_k$.

Proof:

$$U_{n,k} = \begin{pmatrix} U_{k,k} & 1_{k,n-k} \\ 0_{n-k,k} & I_{n-k} \end{pmatrix},$$

$$\pi_k \pi_{n-k} U_{n,k} \sigma_k \sigma_{n-k} = \begin{pmatrix} \pi_k U_{k,k} \sigma_k & 1_{k,n-k} \\ 0_{n-k,k} & \pi_{n-k} \sigma_{n-k} \end{pmatrix},$$

Thus, we can view the action α of $(S_k \times S_{n-k}) \times (S_k \times S_{n-k})$ on W_n^k as composed of two independent actions. One of them is the action of $S_k \times S_k$ on H_k^k (the upper left block) and is actually the regular representation of $S_k \times S_k$, while the second one is an action of $S_{n-k} \times S_{n-k}$ on S_{n-k} (the lower right block) which gives rise to the representation $\bigoplus_{\rho \vdash n-k} S^\rho \otimes S^\rho$. $\square$

This implies the following:

Claim:

$$\chi_{\omega_{n,k}}(\pi_k\pi_{n-k},\sigma_k\sigma_{n-k}) = $$

$$= \begin{cases} 0 & \pi_k \neq e \text{ or } \sigma_k \neq e\,. \\ 0 & \pi_{n-k} \nsim \sigma_{n-k} \in S_{n-k}\,. \\ (k!)^2|C^{n-k}_{\pi_{n-k}}| & \pi_k = \sigma_k = e \text{ and} \\ & \pi_{n-k} \sim \sigma_{n-k} \in S_{n-k} \end{cases}$$

Theorem:

$$\alpha_{H_n^k} = \omega_{n,k} \uparrow_{(S_k\times S_{n-k})\times(S_k\times S_{n-k})}^{S_n\times S_n} .$$

Proof: Write $G = S_n \times S_n$ and $H = (S_k \times S_{n-k}) \times (S_k \times S_{n-k})$ and identify G/H with a prescribed set of left transversals of H in G.

By the definition of W_n^k we have $H \bullet W_n^k = W_n^k$ and therefore, the space $span_{\mathbb{C}} W_n^k$ is invariant the $\bullet$-action (which is exactly the action α) of H.

We clearly have:

$$\begin{aligned} H_n^k &= \{g \bullet U_{n,k} \mid g \in S_n \times S_n\} \\ &= \{(\sigma h) \bullet U_{n,k} \mid \sigma \in G/H, h \in H\} \\ &= \{\sigma \bullet (h \bullet U_{n,k}) \mid \sigma \in G/H, h \in H\} \\ &= \{\sigma \bullet W_n^k \mid \sigma \in G/H\} \\ &= \biguplus_{\sigma \in G/H} \sigma \bullet W_n^k \ , \end{aligned}$$

where $\uplus$ denotes disjoint union.

This implies that

$$\begin{aligned} \alpha_{H_n^k} &= \bigoplus_{\sigma \in G/H} \sigma \bullet span_{\mathbb{C}} W_n^k \\ &= \omega_{n,k} \uparrow_{(S_k \times S_{n-k}) \times (S_k \times S_{n-k})}^{S_n \times S_n} \ , \end{aligned}$$

as claimed. □

Theorem:

$$m\left(\left(\lambda,\mu\right),\alpha_{H_n^k}\right)=\langle\chi_\lambda\downarrow^{S_n}_{S_{n-k}},\chi_\mu\downarrow^{S_n}_{S_{n-k}}\rangle\ ,$$

or, in other words,

$$\alpha_{H_n^k}=\bigoplus_{\lambda,\mu\vdash n}\langle\chi_\lambda\downarrow^{S_n}_{S_{n-k}},\chi_\mu\downarrow^{S_n}_{S_{n-k}}\rangle S^\lambda\otimes S^\mu .$$

The proof uses our above propositions and the Frobenius reciprocity formula.

The number $\langle \chi_\lambda \downarrow^{S_n}_{S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_{n-k}} \rangle$ has a combinatorial interpretation. It follows from the branching rule that this is just the number of ways to delete k boundary cells from the diagrams corresponding to the partitions λ and μ to get the same Young diagram of $n-k$ cells.

In particular:

$\langle \chi_\lambda \downarrow^{S_n}_{S_{n-k}}, \chi_\mu \downarrow^{S_n}_{S_{n-k}} \rangle = 0$ when $|\lambda \triangle \mu| > 2k$ and it does not vanish otherwise.

Corollary:

$m\left((\lambda,\mu), \alpha_{H_n^k}\right) = 0$ when $|\lambda \triangle \mu| > 2k$ and
$m\left((\lambda,\mu), \alpha_{H_n^k}\right) \neq 0$ when $|\lambda \triangle \mu| \leq 2k$.

Use the fact that

$$\beta_{H_n^k} = \alpha_{H_n^k} \downarrow_{S_n}^{S_n \times S_n}$$

to obtain some asymptotic relations.

For $\lambda, \mu, \nu \vdash n$, denote

$$\gamma_{\lambda\mu\nu} = \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi)\chi_\mu(\pi)\chi_\nu(\pi).$$

Easy to see that

$$S^\lambda \uparrow_{S_n}^{S_n \times S_n} \cong \bigoplus_{\mu,\nu \vdash n} \gamma_{\lambda\mu\nu} S^\mu \otimes S^\nu.$$

Remark: The numbers $\gamma_{\lambda\mu\nu}$ appear in the context of the Schur functions within the following formula:

$$s_\lambda(xy) = \sum_{\mu,\nu} \gamma_{\lambda\mu\nu} s_\mu(x) s_\nu(y),$$

where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and (xy) means the set of variables $x_i y_j$ and s_λ, s_μ and s_ν are the Schur functions corresponding to λ, μ and ν respectively..

$$
\begin{aligned}
m\left(\lambda, \beta_{H_n^k}\right) &= \langle \chi_{\beta_{H_n^k}}, \chi_\lambda \rangle = \langle \chi_{\alpha_{H_n^k}} \downarrow_{S_n}^{S_n \times S_n}, \chi_\lambda \rangle \\
&= \langle \chi_{\alpha_{H_n^k}}, \chi_\lambda \uparrow_{S_n}^{S_n \times S_n} \rangle \\
&= \sum_{\mu,\nu \vdash n} \langle \chi_\mu \downarrow_{S_{n-k}}^{S_n}, \chi_\nu \downarrow_{S_{n-k}}^{S_n} \rangle \gamma_{\lambda\mu\nu} \ .
\end{aligned}
$$

Now, using the above asymptotic for $m\left(\lambda, \beta_{H_n^k}\right)$ we get the following

Proposition:

For any $0 < \varepsilon < 1$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} \leq \delta(\varepsilon)$,

$$1 - \varepsilon < \frac{\sum_{\mu,\nu \vdash n} \langle \chi_\mu \downarrow_{S_{n-k}}^{S_n}, \chi_\nu \downarrow_{S_{n-k}}^{S_n} \rangle \gamma_{\lambda\mu\nu}}{(n)_k f^\lambda} < 1 + \varepsilon.$$

Substituting in the above proposition $k = 0$ and $k = n$, we get the following:

$$1 - \varepsilon < \frac{\sum_{\mu \vdash n} \gamma_{\lambda\mu\mu}}{f^\lambda} < 1 + \varepsilon,$$

$$1 - \varepsilon < \frac{\sum_{\mu,\nu \vdash n} \gamma_{\lambda\mu\nu} f^\mu f^\nu}{n! f^\lambda} < 1 + \varepsilon.$$

Remark:

The first statement follows from Theorem R1 and the equality

$$\sum_{\mu \vdash n} \gamma_{\lambda\mu\mu} = \sum_{C \in \hat{S}_n} \chi_\lambda(C)$$

which itself follows from the character orthogonality relations.

Final remark:

Other versions of this work are obtained by considering the actions α and β on certain subsets of the colored permutation groups.