

On Modular Invariants of Finite Groups

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Notation

G **finite group**

V a finite dimensional $\mathbb{K}G$ - module,

V^* dual module with basis x_1, \dots, x_n ,

$A = \text{Sym}(V^*) \cong \mathbb{K}[x_1, \dots, x_n]$, polynomial ring;

$A^G := \{a \in A \mid g(a) = a, \forall g \in G\}$, (\mathbb{N}_0 – graded)

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Example:

$G := \Sigma_n$, symmetric group, acting on $V \cong V^* = \bigoplus_{i=1}^n \mathbb{K}x_i$;
(natural representation)

$A^G = \mathbb{K}[e_1, \dots, e_n]$, symmetric polynomials: generated by elementary symmetric functions e_1, \dots, e_n .

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As in representation theory:

$\text{char } \mathbb{K} \nmid |G| \iff$: non – modular case;

$\text{char } \mathbb{K} \mid |G| \iff$: modular case.

Constructive Aspects

Definition: (Degree bounds, Noether - number)

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- - Noether's proofs **do not work** for $\text{char } \mathbb{K} \nmid |G|$ in general.
 - the Noether bound **does not hold** if $\text{char } \mathbb{K}$ divides $|G|$.
 - Generalization to $\text{char } \mathbb{K} \nmid |G|$
(Fl. (2000), Fogarty/Benson, (2001)).

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Example: $A^G := \mathbb{F}_p[V_{reg}]^{\mathbb{Z}/p}$, $p \geq 5$.

Modular Obstructions

- Transfer map

$$t_1^G : A \rightarrow A^G : a \rightarrow \sum_{g \in G} g(a).$$

Surjective in the non-modular case, but

$t_1^G(A) \trianglelefteq A^G$ **proper** ideal

in the **modular** case.

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$$t_1^G(A) \not\subseteq A^G \text{ proper ideal}$$

in the **modular** case.

- Combinatorics to reduce degrees of generators eg.:

$$x_1 \dots x_m = \frac{(-1)^m}{m!} \sum_{I \subseteq \{1, \dots, m\}} (-1)^{|I|} \left(\sum_{i \in I} x_i \right)^m$$

(needed for $m = 1, \dots, |G|$, so **fails** in modular case).

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but $\mathcal{X} = (x_2 \dots x_k)^+ x_1 + (x_1 x_3 \dots x_k)^+ y_2 - (x_3 \dots x_k)^+ x_1 y_2$,
is a decomposition in **Hilbert ideal** $A^{G,+} \cdot A \triangleleft A$.

Relative Noether Bound

Theorem (Fl., Knop, Sezer, (indep.)):

If $H \leq G$, $|G| \in \mathbb{K}^*$, or $H \trianglelefteq G$ with $[G : H] \in \mathbb{K}^*$, then

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Sketch of proof: $G := \bigoplus_{i=1}^n g_i H$;

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Let $b_1, b_2, \dots, b_n \in A^H$. For fixed \mathbf{i} :

$$\prod_{j=1}^n (g_{\mathbf{i}}(b_j) - g_j(b_j)) = 0.$$

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$$t_H^G(bb_1 \cdots b_n) =$$

$$\sum_{I < \{1, 2, \dots, n\}} (-1)^{n-|I|+1} \left[\prod_{\ell \notin I} g_\ell(b_\ell) \right] \cdot t_H^G(b \cdot \prod_{j \in I} b_j),$$

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Application of t_1^G or t_H^G yields decomposition in A^G .

Known modular degree bounds

$n := \dim(V)$:

- $n(|G| - 1) + |G|^{n2^{n-1}+1} \cdot n^{2^{n-1}+1}$, (Derksen - Kemper)
- $\frac{q^n - 1}{q - 1} \cdot (nq - n - 1)$, (Karaguezian - Symonds),

here $q = |\mathbb{K}|$.

- For permutation representations $G \leq \Sigma_n$:

$$\beta(A^G) \leq \binom{n}{2}. \quad (\text{M Goebel 1996}).$$

Degree bound conjectures

- 1 If $p \nmid [G : H]$, then $\beta(A^G) \leq \beta(A^H) \cdot [G : H]$
(i.e. $H \trianglelefteq G$ not needed).

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- Conjecture 1 true, if $[G : H]! \in \mathbb{K}^*$.
Conjectures 2. and 3. have been proven for p - permutation representations (Fl. 2000, Fl. - Lempken 1997).

The $2p - 3$ conjecture

Let $\text{char}(\mathbb{K}) = p > 3$, $g = (1 \ 2 \ \cdots \ p) \in \Sigma_p$, $G = \langle g \rangle \cong \mathbb{Z}/p$, acting on $A := \mathbb{K}[x_1, \dots, x_p]$,
(i.e. $A = \text{Sym}(V_{reg})$ regular module).

Know:

$$\beta(A^G) \leq \frac{p(p-1)}{2}.$$

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MAGMA - calculations for $p = 5, 7, 11, 13$ showed:

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Expansion of $\prod_{j=1}^{p-1} (\ g^i(b_j) - b_j \)$ and summation over $g^i \in G$ leads to reduction of transfers $t_1^G(b_1 b_2 \cdots b_{p-1})$ in Hilbert - ideal:

The $2p - 3$ theorem

Theorem I [P F., M Sezer, R J Shank, C F Woodcock, 2005]

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Noether numbers (=exact degree bounds) for arbitrary finite $\mathbb{K}G$ -modules:

for example if $W \in \mathbb{K}[G]\text{-mod}$ with indec. summand $V_k \mid W$ of dimension $k > 3$,

(e.g. if $W = V_k \oplus \cdots \oplus V_k$ with diagonal action):

$$\beta(A^G) = (p - 1)\dim(W^G) + p - 2.$$

The Noether Numbers for \mathbb{Z}/p

Theorem II [P F., M Sezer, R J Shank, C F Woodcock, 2005]

Let W be arbitrary finite $\mathbb{K}G$ - module without trivial summands, i.e. $A = \text{Sym}(W)$, $V_k :=$ indecomposable of dimension k . Then

(A) If $V_k \mid W$ for $k > 3$, then

$$\beta(A^G) = (p - 1)\dim(W^G) + p - 2.$$

(B) If $W \cong mV_2 + \ell V_3$, $\ell > 0$, then

$$\beta(A^G) = (p - 1)\dim(W^G) + 1.$$

(C) If $W \cong mV_2$, then

$$\beta(A^G) = (p - 1)\dim(W^G).$$

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- Noether bound for **invariant field** in arbitrary characteristic i.e.

$$\text{Quot}(A^G) = \text{Quot}(A_{\leq |G|}^G).$$

- For arbitrary G , $\exists 0 \neq c := t_1^G(f)$ in degree $\leq |G|$ such that

$$\beta(A_c^G) \leq 2|G| - 1,$$

If $\deg(c) = 1$, then $\beta(A_c^G) \leq |G|$.

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Lemma: The following are equivalent:

- i) $\mathcal{C}(B, A^G) \neq 0$.
- ii) $\text{Quot}(B) = \text{Quot}(A^G)$ and A^G is integral over B .

Method to construct A^G

- Pick "easy" constructible subalgebra $B \leq A^G$, such that $\exists 0 \neq c \in \mathcal{C}(B, A^G)$ (i.e. $c \in B$ with $cA^G \leq B$.)

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- It follows $A^G = B[f_1/c, \dots, f_s/c]$.

Calculation of $cA \cap B$

- Let $A = \mathbb{K}[X_1, \dots, X_n]$, $B := \mathbb{K}[b_1, \dots, b_m] \leq A$,
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- consider polynomial ring $T := \mathbb{K}[X_1, \dots, X_n, Y_1, \dots, Y_m]$,
with ideal $J := (c, b_j - Y_j) \trianglelefteq T$;

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which can be computed , using Groebner bases, without knowing A^G .

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- Substitution $Y_j \mapsto b_j$ gives

$$cA^G = cA \cap B = (\chi_1(b_1, \dots, b_m), \dots, \chi_\ell(b_1, \dots, b_m)).$$

Use of Diagonal-invariants of Σ_n

Let $|G| = n, V \in \mathbb{K}G - \text{mod}$ of dimension k ;

$V_{\downarrow 1} \uparrow^G := \mathbb{K}G \otimes_{\mathbb{K}} V$, the reduced - induced module.

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- There is canonical $\mathbb{K}G$ -epimorphism

$$V_{\downarrow 1}^{\uparrow G} \rightarrow V, g \otimes v \mapsto g(v)$$

which extends to G - equivariant epimorphism of rings:
(called the **Noether - homomorphism**):

$$\nu : B := \text{Sym}(V_{\downarrow 1}^{\uparrow G}) \rightarrow A := \text{Sym}(V).$$

$$\nu(B^G) \leq A^G$$

Use of Diagonal-invariants of Σ_n

- $B = \text{Sym}(V_{\downarrow 1}{}^{\uparrow G}) \cong \text{Sym}(V^{\oplus n}) \cong$

$$\mathbb{K}[X_{11}, \dots, X_{k1}, \dots, X_{1n}, \dots, X_{kn}]$$

G - action on B factors through the Cayley - homomorphism
 $G \hookrightarrow \Sigma_G \cong \Sigma_n$ and Σ_n - action on B defined by

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$B^{\Sigma_n} =:$ ring of (k - fold) “**vector invariants**” of Σ_n .

Noether - homomorphism yields:

$$\mathcal{N} := \nu(B^{\Sigma_n}) \leq A^G$$

with **equality** if $\text{char}(\mathbb{K}) \nmid |G|$.

Vector Invariants of Σ_n

Theorem(H Weyl, D Richman)

If $p = \text{char}(\mathbb{K}) > n$, then

$$\beta(B^{\Sigma_n}) = n.$$

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Theorem (Fl.,Kemper,Woodcock, 2005)

$$\beta(\text{Quot}(B^{\Sigma_n})) = n.$$

Vector Invariants of Σ_n

Recent progress on B^{Σ_n} :

- Fl. (1997):

$$\beta(B^{\Sigma_n}) \leq \max\{n, k \cdot (n - 1)\}$$

with equality if $n = p^s$ and $\text{char } \mathbb{K} = p$.

- E Briand (2004):

Classification of when B^{Σ_n} is generated by the elementary multisymmetries.

- F Vaccarino (2005), D Rydh (2006):

Minimal sets of generators for B^{Σ_n} .

\mathcal{N} in the modular case

Recall: $\mathcal{N} = \nu(B^{\Sigma_n}) \leq A^G$.

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Theorem(Fl.,Kemper,Woodcock (2004))

- $\beta(\mathcal{N}) \leq \max\{|G|, \text{Dim}(A^G) \cdot (|G| - 1)\}$.
- $A^G = \sqrt[p^r]{\mathcal{N}}$ is purely inseparable over \mathcal{N}
- $0 \neq t_1^G(A) \leq \mathcal{C}(\mathcal{N}, A^G) := \{c \in A^G \mid cA^G \subseteq \mathcal{N}\} \leq \mathcal{N}$,
in particular

$$\text{Quot}(\mathcal{N}) = \text{Quot}(A^G);$$

- $\forall 0 \neq c \in t_1^G(A)$:

$$A^G = c^{-1}(cA \cap \mathcal{N}).$$

Back to structural aspects

Question: How close is A^G to being CM ?

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- length of maximal regular sequence in A^G ;
- cohomological co - dimension as module over parameter subalgebra \mathcal{F} :

$$\text{proj.dim } {}_{\mathcal{F}} A^G + \text{depth}(A^G) = \text{Dim}(A^G)$$

(Auslander-Buchsbaum-formula)

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$$\text{depth}(A^G) = \text{Dim}(A^G) \iff A^G \text{ CM.}$$

Depth and Grade

For ideal $I \trianglelefteq A^G$: **grade**(I, A^G) :=
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Theorem (Fl., Shank 2000)

Let $P \leq G$ a Sylow p -group, V^P the P -fix point space
with corresponding (prime) ideal $\mathcal{I} \trianglelefteq A = \text{Sym}(V^*)$.
Then for $\mathfrak{i} := \mathcal{I} \cap A^G \in \text{Spec}(A^G)$:

$$\text{depth } A^G = \text{grade}(\mathfrak{i}, A^G) + \dim V^P.$$

Connection to group cohomology

(joint work with G Kemper and RJ Shank)

Let $h := \text{height of } \mathfrak{i} = \text{codim } V^P$ and

$$c := \min \{k > 0 \mid H^k(G, A) \neq 0\}$$

=: cohomological connectivity

Theorem 1

- $\text{depth}(A^G) \geq \min\{\dim V^P + c + 1, \dim V\}.$
- equality, if $\mathfrak{i} \cdot \alpha = 0$ for some $0 \neq \alpha \in H^c(G, A).$

call A^G **flat**, if equality holds here.

Criterion for flatness

$$P \in \text{Syl}_p(G); c := \min \{k > 0 \mid H^k(G, A) \neq 0\}$$

If $\exists \alpha \in H^c(G, A)$ such that

1. $\alpha \neq 0$;
2. $\text{res}_{G \rightarrow Q}(\alpha) = 0, \forall Q < P$;

then A^G is flat.

- **Theorem**(Fl.+J Elmer, (2007)):

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\text{char}(\mathbb{K}) = 2$.

Then for every non-projective indecomposable $\mathbb{K}[G]$ -module V the ring $\text{Sym}(V)^G$ is flat.