A Conjecture of Foulkes II

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Recap and history

We define H^λ to be the $\mathbb{C}$-vector space whose basis $\mathcal{H}^\lambda$ is the set of all unordered λ-partitions of $\{1, 2, \ldots, n\}$.

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Example:

$$H^{(4,3,3)} \ni \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \cline{1-3} 8 & 9 & 10 & \\ \cline{1-3} \end{array} \quad , \quad \begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & 9 & 10 & \end{array} \in H^{(4,3,3)'} \quad .$$

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Definition: We define the *Wagner map* $\psi_\lambda : H^\lambda \to H^{\lambda'}$ by

$$\{\bar{t}\}\psi_\lambda = \sum_{g \in G_{\{\bar{t}\}}} \{\bar{t}\}' g \quad .$$

Example:

$$\begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \cline{1-3} 8 & 9 & 10 & \\ \cline{1-3} \end{array} \xrightarrow{\psi_{(4,3,3)}} \sum_{g \in G_{\{t\}}} \begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ \cline{4-4} 5 & 6 & 7 & \multicolumn{1}{c}{} \\ 8 & 9 & 10 & \multicolumn{1}{c}{} \end{array} \; g \; .$$

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Independantly Siemon's and Wagner (1986) and Stanley (2000) conjectured the following:

Conjecture(SWS): Let $\lambda \geq \lambda'$. Then the map ψ_λ is injective.

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Coker (1993), Doran (1998), Dent (2000) and Pylyavskyy (2004) have all proved that $\psi_{(b^2)}$ is injective.

Theorem(CDDP): The map $\psi_{(b^2)}$ is injective.

In 2004 Pylyavskyy was the first to show that ψ_λ may not be injective by showing that $H^{(6,2,2,1,1)}$ has larger dimension than $H^{(6,2,2,1,1)'}$.

$$[(6,2,2,1,1)] = \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & & & & \\ \circ & \circ & & & & \\ \circ & & & & & \\ \circ & & & & & \end{array} .$$

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He then reformulated the SWS conjecture to the following:

Conjecture: The map ψ_λ has maximal rank.

In 2006 Sivek showed that Pylyavskyy's conjecture fails by showing (using a computer program) that $\psi_{(4,3,3)}$ does not have maximal rank. He also proved the following:

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Sivek's Lemma: Let λ be obtained by adding a row to μ. Suppose ψ_μ does not have maximal rank. Then ψ_λ does not have maximal rank.

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Example: The map $\psi_{(4,3,3)}$ does not have maximal rank. So $\psi_{(6,4,3,3)}$ does not have maximal rank:

$$[(4,3,3)] = \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \\ \circ & \circ & \circ & \end{array} \quad , \quad [(6,4,3,3)] = \begin{array}{cccccc} \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & & \\ \circ & \circ & \circ & & & \\ \circ & \circ & \circ & & & \end{array} \quad .$$

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Theorem 1.1: Let μ be a partition. Let λ be a partition obtained by adding a "good" column to the front of μ. Suppose the Wagner map of μ is injective. Then the Wagner map of λ is injective.

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Example 1: (5^3) is injective if (3^3) is injective.

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{array} \longrightarrow \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{array} \longrightarrow \begin{array}{ccc} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{array} .$$

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Theorem 1.2: There exists an injective map $H^{(b^a)} \longrightarrow H^{(a^b)}$ when $a \leq 4$ and $b \geq a$.

Example 2: $(5,3,3)$ is injective if $(3,1,1)$ is injective.

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & & \\ \circ & \circ & \circ & & \end{array} \longrightarrow \begin{array}{cccc} \circ & \circ & \circ & \circ \\ \circ & \circ & & \\ \circ & \circ & & \end{array} \longrightarrow \begin{array}{ccc} \circ & \circ & \circ \\ \circ & & \\ \circ & & \end{array} .$$

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Definition: The *twist group* S_{λ^*} of $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the set of $g \in S_r$ such that $\lambda_i = \lambda_{ig}$.

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Example: The twist group of $(4, 3, 3)$ is S_2 and the twist group of $(6, 6, 2, 2, 2)$ is $S_2 \times S_3$.

The twist group acts on M^λ by permuting rows in the natural way:

$$\begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \cline{1-3} 8 & 9 & 10 & \\ \cline{1-3} \end{array} \xrightarrow{(2,3)^*} \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 8 & 9 & 10 & \\ \cline{1-3} 5 & 6 & 7 & \\ \cline{1-3} \end{array} \quad .$$

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Compositions of Homomorphisms

Let $\psi : M^\lambda \to M^\nu$ be defined by $\{t_\lambda\} \mapsto \sum_{g \in G_{\{t_\lambda\}}} \{t_\nu\} g$.

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$$
\begin{array}{ccccc}
M^\lambda & \xrightarrow{\phi} & M^\mu & \xrightarrow{\theta} & M^\nu \\
\{t_\lambda\} & \longrightarrow & \sum_{g \in G_{\{t_\lambda\}}} \{t_\mu\} g & & \\
 & & \{t_\mu\} & \longrightarrow & \sum_{h \in G_{\{t_\mu\}}} \{t_\nu\} h
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Lemma: Suppose every row of $\{t_\mu\}$ is a subrow of $\{t_\lambda\}$ or $\{t_\nu\}$. Then ψ is a scalar multiple of $\phi \circ \theta$.

Example 1: The Wagner map $\psi_\lambda : M^{\lambda} \to M^{\lambda'}$.

$$
\begin{array}{ccccc}
M^{\lambda} & \xrightarrow{\phi} & M^{\lambda^1} & \xrightarrow{\theta} & M^{\lambda'} \\
\{t\} & \longrightarrow & \sum_{g \in G_{\{t\}}} \{t\}^1 g & & \\
 & & \{t\}^1 & \longrightarrow & \sum_{g \in G_{\{t\}^1}} \{t\}' g
\end{array}
$$

$$
\begin{array}{cccc}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & 8 \\
\hline
9 & 10 & 11 & 12 \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|c|ccc}
\cline{2-4}
1 & 2 & 3 & 4 \\
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Lemma: The map ψ_λ is a scalar multiple of the composition $\phi \circ \theta$.

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Lemma: The map ψ_λ is a scalar multiple of the composition $\phi \circ \theta$.

Lemma: Let μ be the partition obtained by removing the left most column of λ. Then the map $\theta|$ is injective iff $\psi_\mu|$ is injective.

Example 2: The map $\phi : M^{\lambda} \to M^{\lambda^1}$.

$$\begin{array}{ccccccc}
M^{\lambda} & \xrightarrow{\epsilon_0} & M^{\lambda^{(1)}} & \xrightarrow{\epsilon_1} & M^{\lambda^{(2)}} & \xrightarrow{\epsilon_2} & M^{\lambda^1} \\
\{t\} & \to & \sum_{g_0 \in G_{\{t\}}} \{t\}^{(1)} g_0 & & & & \\
 & & \{t\}^{(1)} & \to & \sum_{g_1 \in G_{\{t\}}} \{t\}^{(2)} g_1 & & \\
 & & & & \{t\}^{(2)} & \to & \sum_{g_2 \in G_{\{t\}}} \{t\}^{1} g_2
\end{array}$$

$$\begin{array}{cccc}
\hline
1 & 2 & 3 & 4 \\
\hline
5 & 6 & 7 & 8 \\
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\hline
\end{array}
\to
\begin{array}{c|c|ccc}
\hline
\multicolumn{2}{c}{1} & 2 & 3 & 4 \\
\hline
\multicolumn{2}{c}{5} & 6 & 7 & 8 \\
\hline
 & 9 & 10 & 11 & 12 \\
\cline{3-5}
\end{array}
\to
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\multicolumn{1}{c}{1} & 2 & 3 & 4 \\
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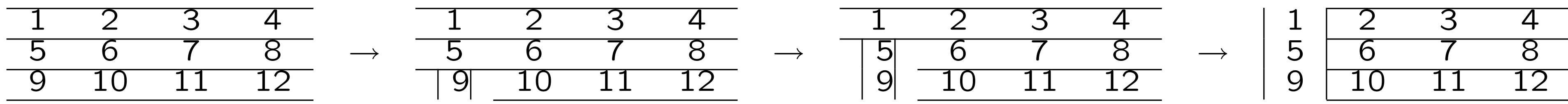
$$

Theorem(Livingstone-Wagner 1965): The map $\epsilon_i : M^{\lambda^{(i)}} \to M^{\lambda^{(i+1)}}$ is injective iff $i > \lambda_{i+1}$.



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Corollary: The map ϵ_i is injective if the hook $h_{i,1}$ has arm at least as long as its leg.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \quad \rightarrow \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \quad \rightarrow \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \quad \rightarrow \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}$$

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$$\begin{array}{cccc}1&2&3&4\\5&6&7&8\\9&10&11&12\end{array} \quad\longrightarrow\quad \begin{array}{cccc}1&2&3&4\\5&6&7&8\\9&10&11&12\end{array} \quad\longrightarrow\quad \begin{array}{cccc}1&2&3&4\\5&6&7&8\\9&10&11&12\end{array} \quad\longrightarrow\quad \begin{array}{c|ccc}1&2&3&4\\5&6&7&8\\9&10&11&12\end{array}$$

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Definition: The first column of $[\lambda]$ is *good* if the arm of $h_{i,1}$ is longer than the leg for all i.

A reformulation of SWS

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Example: Let $\lambda = (7^3, 4^2, 2^2)$

$$
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \circ & \circ & \bullet & A_1 \\
\circ & \circ & \circ & \circ & & & & \\
\circ & \circ & \circ & \bullet & A_2 & & & \\
\circ & \circ & & & & & & \\
\circ & \bullet & A_3 & & & & &
\end{array}
$$

Definition: For $i < j$ define $A(i,j)$ to be the unique hook whose arm contains the removable node A_i and whose leg includes the removable node A_j.

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$$A(1,2) = \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & & & \\ \circ & \circ & \circ & \bullet & & & \\ \circ & \circ & & & & & \\ \circ & \circ & & & & & \end{array} \qquad A(1,3) = \begin{array}{ccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ & & & \\ \circ & \bullet & \circ & \circ & & & \\ \circ & \bullet & & & & & \\ \circ & \bullet & & & & & \end{array}$$

Definition: For $i < j$ define $A(i,j)$ to be the unique hook whose arm contains the removable node A_i and whose leg includes the removable node A_j.

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Definition: The hook $A(i,j)$ is *good* if its arm is at least as long as its leg. A partition is *good* if all $A(i,j)$ are good. A partition is *bad* if it is not good.

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Theorem 2.2: Let λ be a partition with at most three parts. Then the following are equivalent:

(i) There exists an injective map $H^\lambda \to H^{\lambda'}$.

(ii) λ is good.

(iii) The Wagner map $\psi_\lambda|$ is injective.

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Definition: Let $\mathcal{H}^\lambda$ denote the **set** of unordered λ-tabloids.

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Example:

$$\begin{array}{cccc} \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 1 & \\ \cline{1-3} 1 & 1 & 2 & \\ \cline{1-3} \end{array} \in \mathcal{H}^{(4,3,3),(8,2)}$$

Lemma: There exists a bijection $\mathcal{O}(\mathcal{H}^\lambda, S_\nu) \leftrightarrow \mathcal{H}^{\lambda,\nu}$.

Lemma: Suppose that there exists an injective map $H^\lambda \to H^{\lambda'}$. Then there exists an injective map $\mathcal{H}^{\lambda,\nu} \hookrightarrow \mathcal{H}^{\lambda',\nu}$ for all partitions ν.

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Proposition: Let $a - b < c$. Then there exists no injective map $H^{(a,b^c)} \to H^{(a,b^c)'}$.

Example: The sets $\mathcal{O}(\mathcal{H}^{4,3,3}, S_{8,2})$ and $\mathcal{O}(\mathcal{H}^{3,3,3,1}, S_{8,2})$.

$$\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & \\ \cline{1-3} 1 & 2 & 2 & \\ \cline{1-3} \end{array} \longrightarrow \begin{array}{|c|c|c|c|} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & 2 & 2 & \end{array} ,$$

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Proof of Theorem 2.1:

$$[(12,11,9,7^3)] = \begin{array}{cccccccccccc} \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \\ \circ & \circ & \circ & \circ & \circ & \circ & \bullet & \bullet & \bullet & & & \\ \circ & \circ & \circ & \circ & \circ & \circ & \bullet & & & & & \\ \circ & \circ & \circ & \circ & \circ & \circ & \bullet & & & & & \\ \circ & \circ & \circ & \circ & \circ & \circ & \bullet & & & & & \end{array}$$

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Recall Sivek's Lemma:

Let λ be obtained by adding a row to μ. Suppose that $\psi_\mu|$ is not injective. Then $\psi_\lambda|$ is not injective.

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Now $\mu = (9, 7^3)$ is not injective by counting orbits as above so $\lambda = (12, 11, 9, 7^3)$ is not injective and we are done.

Four part partitions

Theorem 2.2 says that the Wagner map controls the existance of an injective map $H^{\lambda} \to H^{\lambda'}$ when λ has at most three parts...

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Unfortinatly for four part partitions we have the following:

Proposition: Let $a \geq 2b$ and suppose $\lambda = (a, b, b-1, b-1)$. Then:

(i) The Wagner map of λ has a kernel, and

(ii) There exists an injective map $H^\lambda \to H^{\lambda'}$.