On the Combinatorics of Diagonal Harmonics for Hook Shapes

Ron Adin
Bar-Ilan University

Jeff Remmel
UCSD

Yuval Roichman
Bar-Ilan University

CIEM, Castro Urdiales

October 2007

A. Introduction

A1. The Garsia-Haiman Module $\mathbf{H}_\mu$

Fix an ordering $w_1, \ldots, w_n$ of the squares of a Young diagram $[\mu]$, and let

$$\Delta_\mu(x_1, \ldots, x_n; y_1, \ldots, y_n) :=$$
$$\det\left(x_i^{row(w_j)-1} y_i^{col(w_j)-1}\right)_{i,j}.$$

Let $\mathbf{H}_\mu$ be the vector space of polynomials spanned by all the partial derivatives of

$$\Delta_\mu(x_1, \ldots, x_n; y_1, \ldots, y_n).$$

Example.

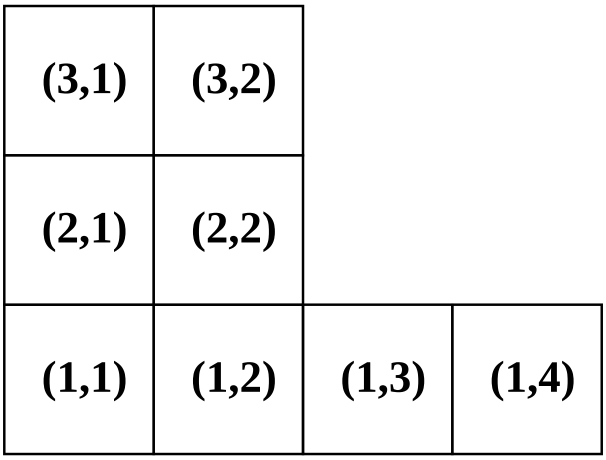

Figure 1: Labelling of the cells of a partition.

Then

$$\Delta_{4,2,2}(x_1, \ldots, x_8; y_1, \ldots, y_8) =$$

$$\det \begin{pmatrix} 1 & y_1 & y_1^2 & y_1^3 & x_1 & x_1 y_1 & x_1^2 & x_1^2 y_1 \\ 1 & y_2 & y_2^2 & y_2^3 & x_2 & x_2 y_2 & x_2^2 & x_2^2 y_2 \\ \vdots & & & & & & & \vdots \\ 1 & y_8 & y_8^2 & y_8^3 & x_8 & x_8 y_8 & x_8^2 & x_8^2 y_8 \end{pmatrix}$$

The n! Theorem

The symmetric group S_n acts on $\mathbf{H}_\mu$ by permuting variables.

Theorem [Haiman '01]

For every partition μ,

$$\mathbf{H}_\mu \cong \mathbf{Q}[S_n]$$

as S_n-modules.

In particular,

$$\dim \mathbf{H}_\mu = n!$$

A2. The Coinvariant Algebra of Type A

Denote $\bar{x} := x_1, \ldots, x_n$.

The symmetric group S_n acts on $\mathbf{Q}[\bar{x}]$ by permuting variables.

Let $\Lambda[\bar{x}]^+$ the subalgebra of symmetric functions without a constant term ;

I_n^+ be the ideal generated by $\Lambda[\bar{x}]^+$.

Theorem [Chevalley '55]

$$\mathbf{Q}[\bar{x}]/I_n^+ \cong \mathbf{Q}[S_n]$$

as S_n-modules.

Note: $\mathbf{Q}[\bar{x}]/I_n^+ \cong \mathbf{H}_{(n)}$ as S_n bi-graded modules.

The Descent Basis

The *descent set* of a permutation $\pi \in S_n$ is

$$\mathrm{Des}(\pi) := \{i \, : \, \pi(i) > \pi(i+1)\}.$$

The *descent monomial* of $\pi \in S_n$ is

$$a_\pi := \prod_{i \in \mathrm{Des}(\pi)} (x_{\pi(1)} \cdots x_{\pi(i)})$$

$$= \prod_{j=1}^{n-1} x_{\pi(j)}^{|\mathrm{Des}(\pi) \cap \{j, \ldots, n-1\}|}.$$

Theorem [Steinberg '75, Garsia-Stanton '84]
The set

$$\{a_\pi \, : \, \pi \in S_n\}$$

forms a basis for the coinvariant algebra of type A.

Decomposition into Irreducibles

Let $SYT(\lambda)$:= the set of all standard Young tableaux of shape λ.

i is a *descent* in a SYT T if $i+1$ lies strictly above and weakly to the left of i.

Denote the set of all descents in T by $\mathrm{Des}(T)$.

The *major index* of T is

$$\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i.$$

Let $m_\lambda^{(k)}$:= the multiplicity of the irreducible S_n-representation S^λ in the k-th homogeneous component of the coinvariant algebra of type A.

Theorem [Lusztig and Stanley '79]

$$m_\lambda^{(k)} = \#\{T \in SYT(\lambda) \,:\, \mathrm{maj}(T) = k\}.$$

A3. An Alternative Description of $\mathbf{H}_\mu$

Consider the inner product $\langle\ ,\ \rangle$ on

$\mathbf{Q}[\bar{x}, \bar{y}] = \mathbf{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$

defined by:

for any $f, g \in \mathbf{Q}[\bar{x}, \bar{y}]$,

$\langle f, g \rangle$ is the constant term of

$$f(\partial_{x_1}, \ldots, \partial_{x_n}; \partial_{y_1}, \ldots, \partial_{y_n})g.$$

Let J_μ be the S_n-module dual to $\mathbf{H}_\mu$

with respect to $\langle\ ,\ \rangle$.

Let

$$\mathbf{H}'_\mu := \mathbf{Q}[\bar{x}, \bar{y}]/J_\mu.$$

B. Garsia-Haiman Modules of Hook Shape

B1. An Explicit Description of J_μ

Theorem. [Aval '00] For $\mu = (k, 1^{n-k})$

the ideal $J_\mu = \mathbf{H}_\mu^\perp$ is generated by

(i) $\Lambda[\bar{x}]^+$ and $\Lambda[\bar{y}]^+$
(the symmetric functions in $\bar{x}$ and $\bar{y}$ without a constant term),

(ii) the monomials

$$x_{i_1} \cdots x_{i_k} \qquad (i_1 < \cdots < i_k),$$

$$y_{i_1} \cdots y_{i_{n-k+1}} \qquad (i_1 < \cdots < i_{n-k+1}),$$

and

(iii) the monomials

$$x_i y_i \qquad (1 \le i \le n).$$

B2. Bases

The k^{th} Descent Basis

Definition. For every $1 \le k \le n$ and $\pi \in S_n$ define

$$d_i^{(k)}(\pi) := \begin{cases} |\mathrm{Des}(\pi) \cap \{i, \ldots, k-1\}|, & \text{if } 1 \le i < k; \\ 0, & \text{if } i = k; \\ |\mathrm{Des}(\pi) \cap \{k, \ldots, i-1\}|, & \text{if } k < i \le n. \end{cases}$$

and the *k^{th} descent monomial*

$$a_\pi^{(k)} :=$$

$$\prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i^{(k)}(\pi)} =$$

$$\prod_{\substack{i \in \mathrm{Des}(\pi) \\ i \le k-1}} (x_{\pi(1)} \cdots x_{\pi(i)}) \cdot \prod_{\substack{i \in \mathrm{Des}(\pi) \\ i \ge k}} (y_{\pi(i+1)} \cdots y_{\pi(n)}).$$

Example. $n = 8$, $k = 4$, and $\pi = 8\ 6\ 1\ 4\ 7\ 3\ 5\ 2$, then $Des(\pi) = \{1, 2, 5, 7\}$,
$(d_1^{(4)}(\pi), \ldots, d_8^{(4)}(\pi)) = (2, 1, 0, 0, 0, 1, 1, 2)$,
and $a_\pi^{(4)} = x_1^2 x_2 y_6 y_7 y_8^2$.

Theorem. For every $1 \le k \le n$, the set

$$\{a_\pi^{(k)} \ : \ \pi \in S_n\}$$

forms a basis for the Garsia-Haiman module $\mathbf{H}'_{(k,1^{n-k})}$.

1^{st} **Proof -** Straightening.

2^{nd} **Proof -** A special case of an inductive basis.

The k^{th} Inversion Number

Definition. For every $1 \le k \le n$ and $\pi \in S_n$ define

$$\mathrm{inv}_i^{(k)}(\pi) :=$$

$$\begin{cases} |\{j \,:\, i < j \le k \text{ and } \pi(i) > \pi(j)\}|, & \text{if } 1 \le i < k; \\ 0, & \text{if } i = k; \\ |\{j \,:\, k \le j < i \text{ and } \pi(j) > \pi(i)\}|, & \text{if } k < i \le n. \end{cases}$$

Example.

If $n = 8$, $k = 4$, and $\pi = 8\ 6\ 1\ 4\ 7\ 3\ 5\ 2$, then
$(inv_1^{(4)}(\pi), \ldots, inv_8^{(4)}(\pi)) = (3, 2, 0, 0, 0, 2, 1, 4)$.

More k^{th} Bases

Define the *k^{th} Artin monomial*

$$b_\pi^{(k)} \quad := \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)}.$$

and the *k^{th} Haglund monomial*

$$c_\pi^{(k)} \quad := \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)}.$$

Theorem.

Each of the following sets :

$$\{a_\pi^{(k)} \ : \ \pi \in S_n\}, \quad \{b_\pi^{(k)} \ : \ \pi \in S_n\}, \quad \{c_\pi^{(k)} \ : \ \pi \in S_n\}$$

forms an (inductive) basis for the Garsia-Haiman module $\mathbf{H}'_{(k,1^{n-k})}$.

B3. Representations

Descent Representations

(= Zigzags)

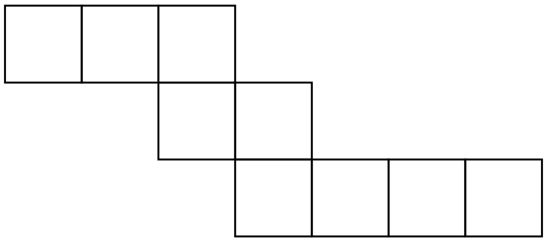

Figure 2: The skew shape corresponding to the composition $(3, 2, 4)$.

The corresponding S_n-representation is called

the $(3, 2, 4)$-zigzag representation

and denoted by $Z^{(3,2,4)}$.

For a composition $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ associate the set

$$\bar{\mathbf{a}} := \{a_1, a_1{+}a_2, a_1{+}a_2{+}a_3, \ldots, a_1{+}a_2{+}\cdots, a_{k-1}\}.$$

For any composition $\mathbf{a}$ of n, the descent class

$$\{\pi \in S_n \;:\; \mathrm{Des}(\pi) = \bar{\mathbf{a}}\}$$

carries the S_n-zigzag representation $Z^{\mathbf{a}}$.

Constructions:

Solomon '66 (Solomon descent representations)

Kazhdan-Lusztig '79

Stanley '82

Gessel '84

Adin-Brenti-Roichman '95

Decomposition of $\mathbf{H}_{(k,1^{n-k})}$ into Descent Representations

Let $\mathbf{H}^{(t_1,t_2)}_{(k,1^{n-k})}$ be the $(t_1,t_2)^{th}$ homogeneous component of $\mathbf{H}_{(k,1^{n-k})}$ (bi-graded by total degrees in the x-s and y-s).

Theorem. For every $t_1, t_2 \geq 0$ and $1 \leq k \leq n$

$$\mathbf{H}^{(t_1,t_2)}_{(k,1^{n-k})} \cong \bigoplus_{\lambda} R^{(k)}_{\lambda},$$

where the sum is over all (n,k)-bipartitions $\lambda = (\mu, \nu)$ with $\mu_{i+1} - \mu_i \in \{0,1\}$ $(\forall i)$, $\nu_{i+1} - \nu_i \in \{0,1\}$ $(\forall i)$ and

$$\sum_{i<k \text{ and } \mu_i > \mu_{i+1}} i = t_1, \qquad \sum_{i \geq k \text{ and } \nu_i < \nu_{i+1}} (n-i) = t_2.$$

Decomposition into Irreducibles

For a standard Young tableau T define

$$\mathrm{maj}_{i,j}(T) := \sum_{\substack{r \in \mathrm{Des}(T) \\ i \le r < j}} r$$

and

$$\mathrm{comaj}_{i,j}(T) := \sum_{\substack{r \in \mathrm{Des}(T) \\ i \le r < j}} (n - r).$$

Theorem [Stembridge '94 + Garsia-Haiman '96]
The multiplicity of the irreducible S_n-representation S^λ in $\mathbf{H}^{(t_1,t_2)}_{(k,1^{n-k})}$ is

$$\#\{T \in SYT(\lambda) \,:\, \mathrm{maj}_{1,k}(T) = t_1,\ \mathrm{comaj}_{k,n}(T) = t_2\}.$$

GRACIAS !

THANKS !

MERCI BEAUCOUP !

TODA RABBA !

The Inductive Basis

Let

$$c \in [n],$$

$$A = \{a_1, \ldots, a_{k-1}\} \subseteq [n] \setminus \{c\},$$

$$\bar{A} := [n] \setminus (A \cup \{c\}).$$

Let

$B_A :=$ a basis of the coinvariant algebra of S_{k-1} acting on $\mathbf{Q}[\bar{x}_A]$, and

$C_{\bar{A}} :=$ a basis of the coinvariant algebra of S_{n-k} acting on $\mathbf{Q}[\bar{y}_{\bar{A}}]$.

Finally define

$$m_{(A,c,\bar{A})} := \prod_{\{i \in A \,:\, i > c\}} x_i \prod_{\{j \in \bar{A} \,:\, j < c\}} y_j \in \mathbf{Q}[\bar{x}, \bar{y}].$$

Then

Theorem. The set

$$\bigcup_{A,c} m_{(A,c,\bar{A})} \; B_A \; C_{\bar{A}} :=$$

$$\bigcup_{A,c} \{ m_{(A,c,\bar{A})} bc \;:\; b \in B_A, c \in C_{\bar{A}} \}$$

forms a basis for the Garsia-Haiman module $\mathbf{H}'_{(k,1^{n-k})}$.

The Inductive Basis - Sketch of the Proof

For every triple $(A, c, \bar{A})$ define an $(A, c, \bar{A})$-permutation $\pi_{(A,c,\bar{A})} \in S_n$

as in the following example:

$n = 9, k = 4, c = 5, A = \{1, 6, 7\}$ then $\pi_{(A,c,\bar{A})} = 761523489$.

Example. For $n = 4$ and $k = 3$, the list of $(A, c, \bar{A})$-permutations is

$$\pi_{(\{34\},2,\{1\})} = 4321,\ \pi_{(\{34\},1,\{2\})} = 4312,\ \ldots,$$

$$\pi_{(\{12\},4,\{3\})} = 2143,\ \pi_{(\{12\},3,\{4\})} = 2134.$$

For a given n and k, order the distinct $(A, c, \bar{A})$-permutations in reverse lexicographic order.

Example (cont.)

$$4321 <_L 4312 <_L 4231 <_L 4213$$

$$<_L 4132 <_L 4123 <_L 3241 <_L 3214$$

$$<_L 3142 <_L 3124 <_L 2143 <_L 2134.$$

Index the permutations and corresponding monomials by order

$$\pi_1 = 4321,\ \pi_2 = 4312,\ \pi_3 = 4231,\ \ldots,$$

$$\pi_{11} = 2143,\ \pi_N = \pi_{12} = 2134.$$

$$m_1 = x_4 x_3 y_1,\ m_2 = x_4 x_3,\ m_3 = x_4 y_1,$$

$$m_{11} = y_3,\ m_N = m_{12} = 1.$$

Let

$$I_0 := J_{(k,1^{n-k})}$$

and define

$$I_t := I_{t-1} + m_t\mathbf{Q}[\bar{x},\bar{y}] \qquad (1 \le t \le N).$$

Clearly,

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_N = \mathbf{Q}[\bar{x},\bar{y}].$$

and

$$\mathbf{H}'_{(k,1^{n-k})} = \mathbf{Q}[\bar{x},\bar{y}]/I_0 \cong \bigoplus_{t=1}^{N}(I_t/I_{t-1})$$

as vector spaces.

Lemma. *The set* $m_t \cdot B_A \cdot C_{\bar{A}}$ *is a basis for* I_t/I_{t-1}.

Proof of Lemma (Sketch).

It is shown that if $i \notin A$ then $m_t x_i \in I_{t-1}$

by a combinatorial analysis of four complementary cases :

Similarly, if $j \notin \bar{A}$ then $m_t y_j \in I_{t-1}$.

This implies that the natural projection

$$f_t : m_t \mathbf{Q}[\bar{x}, \bar{y}] \longrightarrow I_t/I_{t-1}.$$

is well defined on the quotient

$$m_t \mathbf{Q}[\bar{x}, \bar{y}]/$$

$$\left[m_t \cdot \left(\sum_{i \notin A} \langle x_i \rangle + \sum_{j \notin \bar{A}} \langle y_j \rangle + \langle \Lambda[\bar{x}]^+ \rangle + \langle \Lambda[\bar{y}]^+ \rangle \right) \right] \cong$$

$$m_t \cdot \mathbf{Q}[\bar{x}_A]/\langle \Lambda[\bar{x}_A]^+ \rangle \cdot \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle \Lambda[\bar{y}_{\bar{A}}]^+ \rangle).$$

and is essentially an isomorphism.

Decomposition of $\mathbf{H}_{(k,1^{n-k})}$ into Descent Representations

Proof Sketch

Notation.

A pair of partitions $\lambda = (\mu, \nu)$ is called an (n, k)-bipartition if μ has at most $k-1$ parts and ν has at most $n-k$ parts.

Let $\trianglelefteq$ be the dominance order on bipartitions:

$(\mu^1, \nu^1) \trianglelefteq (\mu^2, \nu^2)$ if $\mu^1 \trianglelefteq \mu^2$ and $\nu^1 \trianglelefteq \nu^2$.

For $\pi \in S_n$ and

$$a_\pi^{(k)} = \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i},$$

let

$$\lambda(a_\pi^{(k)}) := ((d_1, d_2, \ldots, d_{k-1}), (d_n, d_{n-1}, \ldots, d_{k+1}))$$

be its *exponent bipartition*.

For an (n, k)-bipartition $\lambda = (\mu, \nu)$ let

$$I_\lambda^{(k)\trianglelefteq} :=$$

$$\mathrm{span}_{\mathbf{Q}}\{a_\pi^{(k)} + J_{(k,1^{n-k})} \,:\, \pi \in S_n,\ \lambda(a_\pi^{(k)}) \trianglelefteq \lambda\},$$

and

$$I_\lambda^{(k)\triangleleft} :=$$

$$\mathrm{span}_{\mathbf{Q}}\{a_\pi^{(k)} + J_{(k,1^{n-k})} \,:\, \pi \in S_n,\ \lambda(a_\pi^{(k)}) \triangleleft \lambda\}$$

be subspaces of the module $\mathbf{H}'_{(k,1^{n-k})}$. Let

$$R_\lambda^{(k)} := I_\lambda^{(k)\trianglelefteq} / I_\lambda^{(k)\triangleleft}.$$

Proposition. $I_\lambda^{(k)\trianglelefteq}$, $I_\lambda^{(k)\triangleleft}$ and thus $R_\lambda^{(k)}$ are S_n-invariant.

Lemma. Let $\lambda = (\mu, \nu)$ be an (n,k)-bipartition. Then

$$R_\lambda^{(k)} \neq \{0\} \iff$$

$$(\forall 1 \leq i < k-1) \quad \mu_i - \mu_{i+1} \in \{0,1\} \text{ and}$$

$$(\forall 1 \leq i < n-k) \quad \nu_i - \nu_{i+1} \in \{0,1\}.$$

If these conditions hold then

$$\{a_\pi^{(k)} + I_\lambda^{(k)\triangleleft} \,:\, \mathrm{Des}(\pi) = A_\lambda\}$$

is a basis for $R_\lambda^{(k)}$, where

$$A_\lambda :=$$

$$\{1 \leq i < n \,:\, \mu_i - \mu_{i+1} = 1 \text{ or } \nu_{n-i} - \nu_{n-i+1} = 1 \,\}.$$

Let $\lambda = (\mu, \nu)$ be an (n, k)-bipartition with $R_\lambda^{(k)} \neq \{0\}$. Recall

$$A_\lambda :=$$

$$\{1 \leq i < n \ : \ \mu_i - \mu_{i+1} = 1 \text{ or } \nu_{n-i} - \nu_{n-i+1} = 1 \ \}.$$

Theorem.

$R_\lambda^{(k)}$ is isomorphic as an S_n-module

to the Solomon descent representation

determined by the descent class

$$\{\pi \in S_n \ : \ \mathrm{Des}(\pi) = A_\lambda\}.$$

An Explicit Formula for the Action

Theorem. The S_n-action on $R_\lambda^{(k)}$ is given by

$$s_j(a_\pi^{(k)}) =$$

$$\begin{cases} a_{s_j\pi}^{(k)}, & |\pi^{-1}(j+1) - \pi^{-1}(j)| > 1; \\ a_\pi^{(k)}, & \pi^{-1}(j+1) = \pi^{-1}(j) + 1; \\ -a_\pi^{(k)} - \sum_{\sigma \in A_j(\pi)} a_\sigma^{(k)}, & \pi^{-1}(j+1) = \pi^{-1}(j) - 1. \end{cases}$$

Example. Let $\pi = 2416573 \in S_7$ and $j = 5$.
Then:

$$\mathrm{Des}(\pi) = \{2, 4, 6\};$$

$$A_j(\pi) = \{24\underline{1756}3,\ 24\underline{5617}3,\ 24\underline{5716}3,\ 24\underline{6715}3\}.$$