Symmetric Functions in Noncommutative variables and MacMahon Symmetric functions

Mercedes Helena Rosas

Universidad de Sevilla, España

Before the beginnings.

* In 1936, Margarete Wolf proves a version for the fundamental theorem of symmetric functions for the situation where the variables do not commute.

She was interested in a theory of symmetric functions where the variables are “completely independent and completely noncommutative”

* In 1972, Peter Doubilet, observes that the partition lattice gives an elegant combinatorial framework for the study symmetric functions.

Outline of the talk

* Combinatorics of set partitions

* Symmetric functions in noncommutative variables.

* The action of the symmetric group

* Symmetric functions

* Young subgroups of the symmetric group

* MacMahon symmetric functions

Combinatorics of set partitions

A *set partition* A of m, written $A \vdash [m]$, is a collection of nonempty subsets $A_1, A_2, \ldots, A_k \subseteq [m] = \{1, 2, \ldots, m\}$ such that $[m]$ is equals to the disjoint union $A_1 \cup A_2 \cup \cdots \cup A_k$.

The number of set partitions is given by the Bell numbers.

$$B_0 = 1 \qquad\qquad B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} B_i$$

The next seven Bell numbers are $1, 2, 5, 15, 52, 203, 877$.

The lattice structure on set partitions

For $A, B \vdash [n]$ that $A \leq B$ if for each $A_i \in A$ there is a $B_j \in B$ such that $A_i \subseteq B_j$ (otherwise stated, that A is finer than B).

The set of set partitions of $[n]$ with this order forms a poset with rank function given by $n - k$ where k the length of the set partition. This poset has minimal element $\{1, 2, \cdots, n\}$ and maximal element $\{12 \cdots n\}$.

The largest element smaller than both A and B will be denoted $A \wedge B = \{A_i \cap B_j : 1 \leq i \leq \ell(A), 1 \leq j \leq \ell(B)\}$ while the smallest element larger than A and B is denoted $A \vee B$.

Let $A = \{138, 24, 5, 67\}$ and $B = \{1, 238, 4567\}$. A and B are not comparable in the inclusion order on set partitions. We calculate that $A \wedge B = \{1, 2, 38, 4, 5, 67\}$ and $A \vee B = \{12345678\}$.

Symmetric functions in noncommutative variables

(I will be following work with Bruce Sagan, and parts of my thesis).

The space of symmetric functions is a subspace of the space polynomials in noncommutative variables.

$$NCSym_n \subseteq \mathbb{Q}\langle X_n \rangle$$

Indeed, it is the space of invariants under the canonical action of the symmetric group.

$$NCSym_n = \mathbb{Q}\langle X_n \rangle^{\mathfrak{S}_n}$$

Monomial symmetric functions in noncommutative variables

Monomial NCSFs corresponding to set partitions of size 3 in a polynomial algebra with 4 variables.

$$
\begin{aligned}
\mathbf{m}_{\{123\}}[X_4] &= x_1x_1x_1 + x_2x_2x_2 + x_3x_3x_3 + x_4x_4x_4. \\
\mathbf{m}_{\{12,3\}}[X_4] &= x_1x_1x_2 + x_1x_1x_3 + x_1x_1x_4 + x_2x_2x_1 + x_2x_2x_3 + x_2x_2x_4 \\
&\qquad x_3x_3x_1 + x_3x_3x_2 + x_3x_3x_4 + x_4x_4x_1 + x_4^2x_2 + x_4^2x \\
\mathbf{m}_{\{13,2\}}[X_4] &= x_1x_2x_1 + x_1x_3x_1 + x_1x_4x_1 + x_2x_1x_2 + x_2x_3x_2 + x_2x_4x_2 \\
&\qquad x_3x_1x_3 + x_3x_2x_3 + x_3x_4x_3 + x_4x_1x_4 + x_4x_2x_4 + x \\
\mathbf{m}_{\{23,1\}}[X_4] &= x_2x_1x_1 + x_3x_1x_1 + x_4x_1x_1 + x_1x_2x_2 + x_3x_2x_2 + x_4x_2x_2 \\
&\qquad x_1x_3x_3 + x_2x_3x_3 + x_4x_3x_3 + x_1x_4x_4 + x_2x_4x_4 + x \\
\mathbf{m}_{\{1,2,3\}}[X_4] &= \sum_{\sigma\in S_4} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}.
\end{aligned}
$$

$\underline{NCSym_n}$

The vector space $NCSym_n$ will be defined as the linear span of the elements

$$\mathbf{m}_A[X_n] = \sum_{\nabla(i_1,i_2,\ldots,i_m)=A} x_{i_1}x_{i_2}\cdots x_{i_m}$$

for $A \vdash [m]$, where the sum is over all sequences with $1 \leq i_j \leq n$.

For the empty set partition, we define by convention $\mathbf{m}_{\{\}}[X_n] = 1$. If $\ell(A) > n$ we must have that $\mathbf{m}_A[X_n] = 0$.

Since for any permutation $\sigma \in S_n$,

$$\nabla(i_1, i_2, \ldots, i_m) = \nabla(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_m)),$$

we also know $\sigma\mathbf{m}_A[X_n] = \mathbf{m}_A[X_n]$.

$\underline{NCSym_n}$

Now let $NCSym_n$ be the space of polynomials of $\mathbb{Q}\langle X_n\rangle$ which are invariant under the action of $\mathfrak{S}_n$.

For any element $f \in NCSym_n$, if $\nabla(i_1, i_2, \ldots, i_k) = \nabla(j_1, j_2, \ldots, j_k)$ then the coefficient of $x_{i_1} x_{i_2} \cdots x_{i_m}$ in f is equal to the coefficient of $x_{j_1} x_{j_2} \cdots x_{j_k}$ in f.

We therefore conclude that $\{\mathbf{m}_A[X_n]\}_{\ell(A)\leq n}$ is a basis for $NCSym_n$.

In addition $NCSym_n$ has a ring structure where the product in this ring is defined as the natural extension of the ring structure on $\mathbb{Q}\langle X_n\rangle$.

The forgetful map :

The forgetful map :

$$\rho : \mathbb{Q}\langle X_n \rangle \longrightarrow \mathbb{Q}[X_n]$$

the map that lets the variables to conmute

What happens to the monomial basis of $NCSym_n$ under the action of the forgetful map ??

The forgetful map :

There is a natural mapping from set partitions to integer partitions given by

$$\lambda(A) = (|A_1|, |A_2|, \ldots, |A_k|),$$

where we assume that the parts of the partition have been listed in weakly decreasing order.

For instance,

$$\lambda(14.256.37.8) = (3, 2, 2, 1)$$

Theorem : The image of the monomial symmetric function under the forgetful map are

$$\rho(\mathbf{m}_A) = A^! m_{\lambda(A)}$$

Symmetric functions

Let $NCSym$ and Sym be the inverse limits of $NCSym_n$ and Sym_n.

The forgetful map induced a map from

$$\rho : NCSym \longrightarrow Sym$$

How is this useful ?

Other basis for Sym :

The power sums.

$$\mathbf{p}_A = \sum_{B \geq A} \mathbf{m}_A$$

Indeed, the $\mathbf{p}_A$ deserve to be called power sums, since

$$\rho(\mathbf{p}_A) = p_{\lambda(A)}$$

Using Möbius inversion, we can write the $\mathbf{m}_A$ in the power sum basis :

$$\mathbf{m}_A = \sum_{B \geq A} \mu(A, B)\mathbf{p}_B$$

The lifting map

The lifting map is a right inverse for the projection map ρ.

$$\tilde{\rho} : Sym \longrightarrow NCSym$$

It is defined by linearly extending

$$\tilde{\rho}(m_\lambda) = \frac{\lambda!}{n!} \sum_{A:\lambda(A)=\lambda} m_A$$

Computing scalar products

We define an scalar product :

$$\langle p_A, p_B \rangle = n! \frac{\delta_{A,B}}{|\mu(\hat{0}, B)|}$$

Theorem: The bilinear form $\langle\ ,\ \rangle$ has the following properties:

* It is symmetric and positive definite, hence it defines a scalar product.

* It is invariant under the action of the symmetric group on *places.*

* It makes the lifting map into an isometry.

Young subgroups of the symmetric group

Let $u = (u_1, u_2, \cdots, u_k)$ be a vector in $\mathbb{N}^k$ whose coordinates add to n.

Then, $\mathfrak{S}_u$ denotes the Young subgroup of $\mathfrak{S}_n$,

$$S_{\{1,2,\cdots,u_1\}} \times \cdots \times S_{\{n-u_k+1,\cdots,n\}}$$

The action of S_u partition $[n]$ into equivalence classes that we order using the smallest element in each.

The type of a set partition $B_1.B_2.\cdots.B_\ell$ under the action of S_u is the vector partition

$$\lambda = \lambda_1\lambda_2\cdots\lambda_l$$

where λ_k is the vector whose i^th coordinate is the number of elements of B_i in the i^th coordinate class.

MacMahon symmetric functions

The monomial symmetric functions

Let $A = (a_1, \cdots, a_n) \cdots (c_1, \cdots, c_n)$. Then m_A be the result of symmetrizing

$$x_1^{a_1} y_1^{a_2} \cdots z_1^{a_n} \cdot x_2^{b_1} y_2^{b_2} \cdots z_2^{b_n} \cdots x_r^{c_1} y_r^{c_2} \cdots z_r^{c_n}.$$

The power sum symmetric functions

$$p_{(u_1, u_2, \cdots, u_k)} = \sum_{i \geq 1} y_i^{u_1} x_i^{u_2} \cdots z_i^{u_k}$$

and extend to vector partitions multiplicatively.

The forgetful map and Young subgroups:

There is also a natural mapping from set partitions to vector partitions given by $\lambda(A)$ equals the type of A under $\mathfrak{S}_u$.

We assume that the parts of the partition have been listed in weakly decreasing order.

For instance, if our Young subgroup is $\mathfrak{S}_{(4,2,2)}$. Then the equivalence classes are $1234/56/78$, and

$$\lambda(14.256.37.8) = ((2,0,0),(1,2,0),(1,0,1),(0,0,1)) \vdash (4,2,2)$$

MacMahon symmetric functions and the projection map

Theorem : Let $\mathfrak{S}_u$ be a Young subgroup of the symmetric group $\mathfrak{S}_n$. Let A be a set partition of $[n]$ and let λ be the type of A under $\mathfrak{S}_n$. Then, under the projection map

$$m_A \mapsto \lambda^! m_\lambda \qquad\qquad p_A \mapsto p_\lambda$$

Why this may be useful :

* Representation theory (including Schur functions) for $NCSym$. (with Nantel Bergeron, Hohlweg, R, Zabrocki).

* Generating functions in noncommutative variables are sometimes easier to manipulate.

* There is are two analogues to Chevalley's theorem for the ring of polynomials in noncommutative variables. (Nantel Bergeron, Reutenauer, R, Zabrocki)

* It is possible to compute the graded Frobenius characteristic for the two spaces of Harmonics appearing in the previous decomposition (Briand, R, Zabrocki).