A Conjecture of Foulkes, I

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- The Conjecture
- Maps
- More Conjectures?
- Foulkes' Theorem for $3 = b \leq a$

- Maps between permutation modules
- An application

The Conjecture

Let $0 \leq a, b$ be integers and $N = \{1...n\}$ where $n = a \cdot b$. Let $P(a^b)$ be the **set of all partitions of** N **into** b **parts of size** a, thus

$$|P(a^b)| = \frac{1}{b!} \binom{n}{a}\binom{n-a}{a}...\binom{a}{a} .$$

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$$|P(a^b)| = \frac{1}{b!} \binom{n}{a}\binom{n-a}{a}\ldots\binom{a}{a} .$$

For instance,

$$\begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline \end{array} \quad = \quad \begin{array}{cccc} \hline 9 & 10 & 11 & 12 \\ \hline 6 & 8 & 7 & 5 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array}$$

is an element in $P(4^3)$.

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Conjecture (Foulkes 1950): **If** $b \le a$ **then**

$$\mathsf{mult}\Big(I, \mathbb{C}P(a^b)\Big) \le \mathsf{mult}\Big(I, \mathbb{C}P(b^a)\Big)$$

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This is well-known to hold if partition classes are **ordered**, i.e.

$$\begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline \end{array} \neq \begin{array}{cccc} \hline 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 \\ \hline 9 & 10 & 11 & 12 \\ \hline \end{array} .$$

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The conjecture is true for $b = 2 \leq a$. There are several proofs, see for instance the **Encyclopedia (1981)** by **James & Kerber**, in connection to plethysm. (I will show a different proof later.)

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The conjecture is true for $b = 3 \leq a$ (**Dent & Siemons, 2000**) and I will outline a proof. Recently **Tom McKay, 2007** at UEA proved the same for $b = 4 < a$. You will hear his report after this talk.

In an AMS millenium survey **Richard Stanley (2000)** writes on “Positivity problems and Conjectures in Algebraic Combinatorics”. In this article the Foulkes conjecture appears as an outstanding problem in positivity: If λ, μ are partitions of n,

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Invariant theory: The conjecture also plays some role for the theory of multi-symmetric polynomials. **Brion (1993)** shows that the Foulkes conjecture is true for $a \gg b$. There are also applications in **rational homotopy** and the **homology of suspensions.**

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Apart from what is mentioned above the conjecture is open, as far as I am aware. It has been a challenging playground for ideas in the past 60 year, with the occasional success story.

Maps

In which way could one prove statements of the kind

$$\text{mult}\Big(I,\, \mathbb{C}P(a^b)\Big) \leq \text{mult}\Big(I,\, \mathbb{C}P(b^a)\Big) \text{ ?}$$

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There is little hope to work out the multiplicities, these are known in a very few cases only, and very difficult to control.

Instead, the conjecture is the same as saying that **there exists a $\mathbb{C}G$-homomorphisms**

$$\varphi : \mathbb{C}P(a^b) \longrightarrow \mathbb{C}P(b^a)$$

which is injective. ($G := \text{Sym}_n$ for the remainder.) So, what are the **standard constructions** for such maps?

Consider a simple example. A standard partition in $P(a^b) = P(4^3)$ is

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Since φ is a $\mathbb{C}G$-map we must have $w^g = w$, for all $g \in G_x$. Thus,

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This describes all possible G-homomorphisms

$$\varphi : \ \mathbb{C}P(4^3) \to \mathbb{C}P(3^4).$$

Probably the most natural choice is

$$u = \left|\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right| \,,$$

and for this choice the map is denoted $\varphi^{(4^3)}$. So, this is the **standard map**

$$\varphi^{(4^3)} : \quad \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline \end{array} \quad \longmapsto \quad \sum_{g \in G_x} \left|\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right| \; g$$

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Theorem (Black & List, 1987): If the standard map $\varphi^{(a^a)}$ is injective then the Foulkes conjecture is true for all $b < a$.

Theorem: The Foulkes conjecture is true for all $a \geq b = 2$.

A sketch of Dent's proof (1997): Let $\varphi = \varphi^{(a^2)}$ be the standard map $\varphi : \mathbb{C}P(a^2) \to \mathbb{C}P(2^a)$ and consider the map

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This map is symmetric (as a matrix) and lies in the centralizer algebra of G on $P(a^2)$. This module is multiplicity-free. (It is a submodule of the module of a-element subsets of $\{1, ..., 2a\}$.)

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Therefore one can use technique from association schemes to work out the eigenvalues of $\varphi^T \varphi$. These are

$$\rho_i = (a-i)!\ i!\ 2^{a-2i-1} \binom{2i}{i} \neq 0$$

for $i = 0, 1, ..., \lfloor \frac{a}{2} \rfloor$. $\square$

Moreover, each eigenspace $E_{\rho_i} = E_{\lambda_i}$ is associated to a partition $\lambda_i \trianglerighteq a^2$ in the dominance order, via the corresponding Specht-module.

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The interesting fact in this proof is that

$$\rho_i > \rho_j \quad \textbf{iff} \quad \lambda_i \trianglerighteq \lambda_j.$$

Any More Conjectures?

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r)$ be a partition of n and let $P(\lambda)$ be the set of all **unordered** λ-partitions of $\{1, ..., n\}$. A typical element in $P(\lambda)$ is

$$x = \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & 10 & \\ \hline 11 & & & \\ \hline \end{array} = \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 8 & 9 & 10 & \\ \hline 5 & 6 & 7 & \\ \hline 11 & & & \\ \hline \end{array} .$$

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As before, let $\mathbb{C}P(\lambda)$ be the corresponding permutation module.

Then again there is a **standard map**

$$\varphi^\lambda : \mathbb{C}P(\lambda) \longrightarrow \mathbb{C}P(\lambda')\,,$$

where λ' is the conjugate partition,

define in the analogous fashion:

$$\varphi^\lambda \colon\ x = \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & 10 & \\ \hline 11 & & & \\ \hline \end{array} \quad \longmapsto \quad \sum_{g \in G_\lambda} \left| \begin{array}{c} 1 \\ 5 \\ 8 \\ 11 \end{array} \right| \begin{array}{c} 2 \\ 6 \\ 9 \\ \ \end{array} \left| \begin{array}{c} 3 \\ 7 \\ 10 \\ \ \end{array} \right| \begin{array}{c} 4 \\ \ \\ \ \\ \ \end{array} \left| \begin{array}{c} \mathsf{g} \\ \ \\ \ \\ \ \end{array} \right. .$$

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Conjecture (Wagner & Siemons, 1986; Stanley, 2000): The map $\varphi^\lambda : \mathbb{C}P(\lambda) \longrightarrow \mathbb{C}P(\lambda')$ has maximal rank. (So φ^λ is surjective or injective, and λ' is the conjugate partition.)

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Note: (i) As stated, this includes the Foulkes conjecture.
(ii) The conjecture is true for *ordered partitions*.

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Note: (i) Recall, $\dim \mathbb{C}P(5^5) \approx 5 \cdot 10^{12}$ and computations are difficult.

(ii) Recall that the Foulkes conjecture is true for all $b \leq a \leq 5$, the N&M Theorem has no bearing on this.

(iii) We know that $\varphi^{(2^2)}$, $\varphi^{(3^3)}$ and $\varphi^{(4^4)}$ (Jacob, 2004) all are of maximal rank. This will be crucial later on. Unfortunately no results known for larger a.

Just One More Conjecture (S, 2000):

Let $\varphi = \varphi^\lambda : \mathbb{C}P(\lambda) \to \mathbb{C}P(\lambda')$ be the standard map. Then a minimal eigenvalue $\rho \geq 0$ of $\varphi^T\varphi$ appears on an eigenspace E_ρ containing a Specht module S^μ with μ minimally dominating λ.

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Sketch of Proof : (i) Let μ dominate a^3. Find a lower bound for the multiplicities m_μ of the Specht module S^μ in $\mathbb{C}P(a^3)$ by writing down sufficiently many linearly independent homomorphism $S^\mu \to \mathbb{C}P(a^3)$.

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(ii) Find an upper bound for $\sum m_\mu^2$. This involves counting intersection arrays of partitions, and in particular 3×3 matrices with constant row/column sums. It turns out that the bound is a certain **invariant of "binary seventhics" (Cayley 1879).**

(iii) By now m_λ is known. Write down m_λ linearly independent homomorphisms $S^\lambda \to \mathbb{C}P(3^a)$. $\square$

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We have partial results on the eigenvalues of $\varphi^T\varphi$, corroborating the conjecture earlier for a^3. There is also computational evidence confirming the conjecture for all a^3 with $a \leq 8$. (This requires looking at representations of degree $\simeq 10^{10}$.)

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In the next slide we will see some illustration of this.

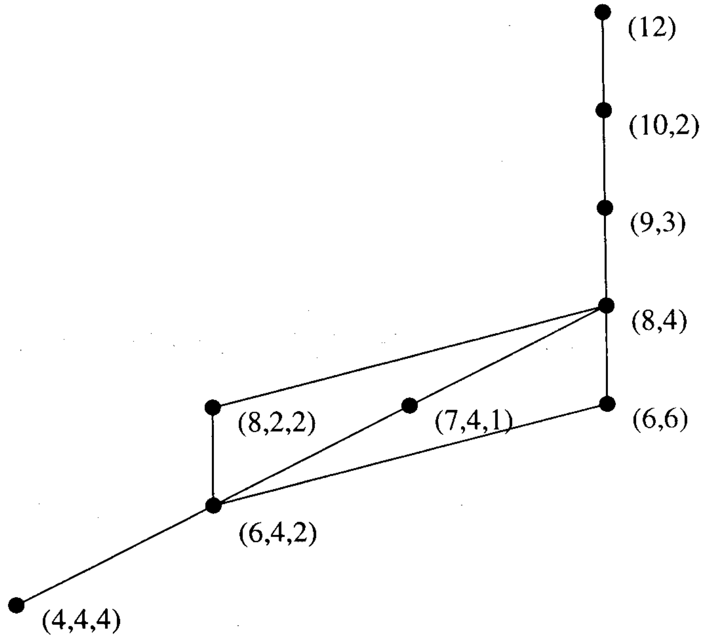

Figure 5: Partial Ordering of the Partitions of 12

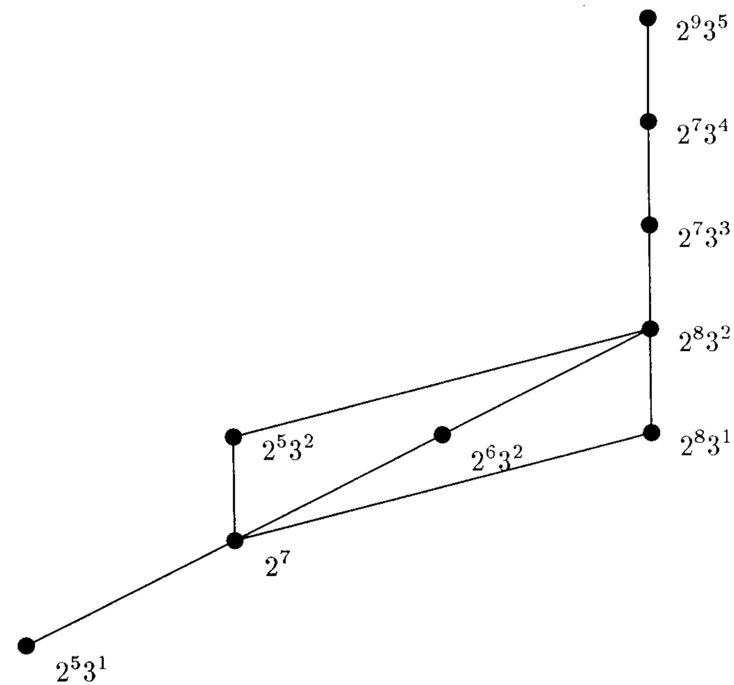

Figure 6: Partial Ordering of the Eigenvalues of $M^{3,4}(M^{3,4})^T$

$P(4^3)$:

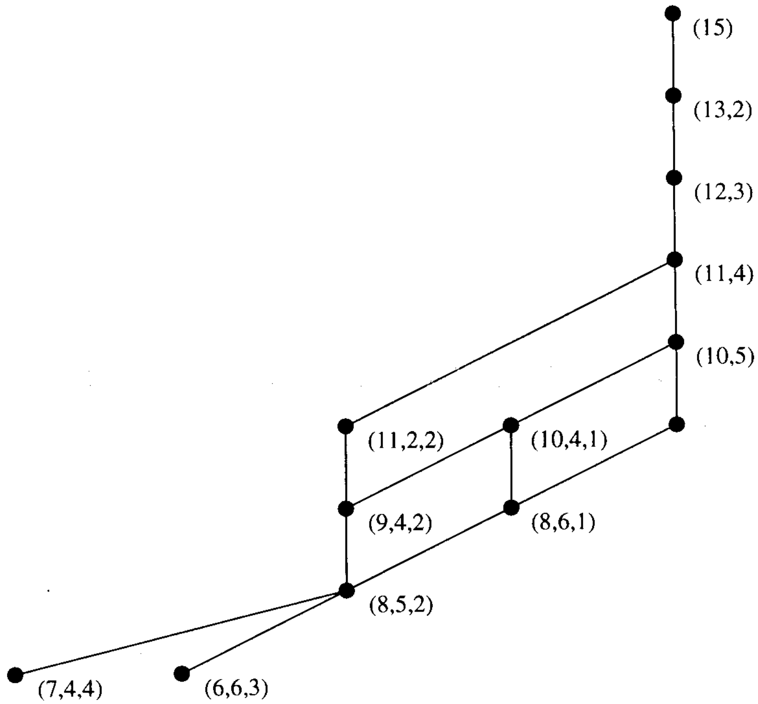

Figure 7: Partial Ordering of the Partitions of

$P(5^3)$:

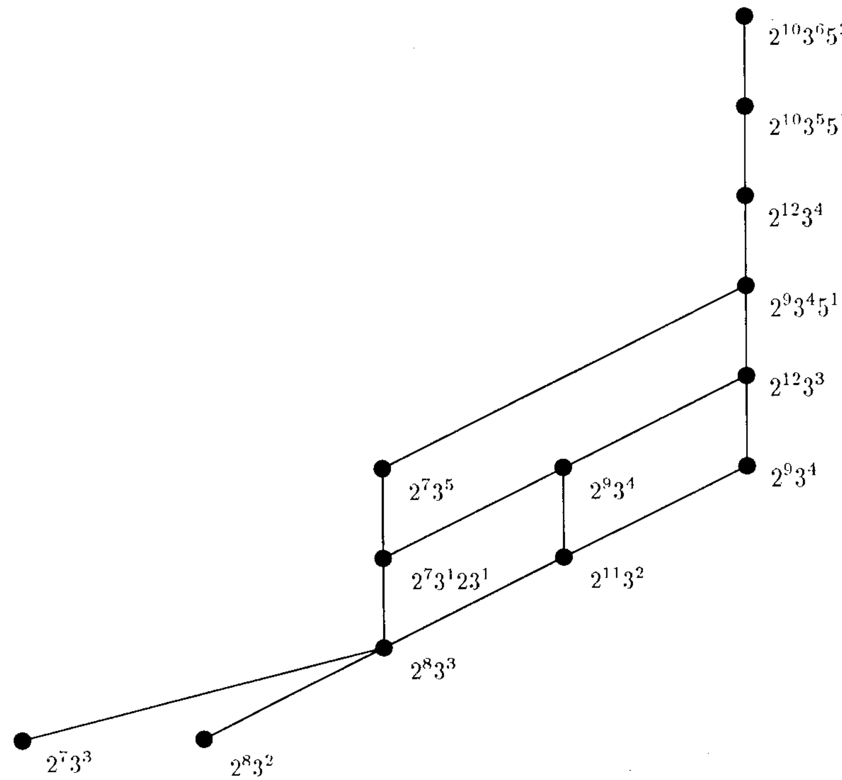

Figure 8: Partial Ordering of the Eigenvalues of $M^{3,5}(M^{3,5})^T$

We have seen two proof variants of Foulkes' Conjecture. In **Tom McKay's** talk

A Conjecture of Foulkes, II

you will see one further proof that generalizes the proofs here.

Permutation homomorphisms

Let G act as a permutation group on a set Ω and also on a set Δ. Let F be a field and suppose there is an **injective** FG-**homomorphism**

$$\varphi : \ F\Omega \ \to \ F\Delta \ .$$

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Here is an example: Let $G = GL(3, q)$ act on the Points and the Lines of the projective plane over GF(q). Then the stabilizer of a point has orbits of length 1 and $q^2 + q$ on Points but orbits of length $q + 1$ and q^2 on Lines.

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The reason for this is a famous lemma of Brauer, 1941:

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One can extend this a little further:

Lemma (Siemons & Zalesskii 2002): Let H be a cyclic group acting on Ω. Then the number of orbits of length $|H|$ is equal to the multiplicity of the regular module FH in $F\Omega$.

Corollary: Let H be a cyclic group acting on the set Ω and on the set Δ. If there exists an injective H-homomorphism $\varphi : F\Omega \to F\Delta$ then the number of H-orbits of length $|H|$ on Δ is no less than the number of such orbits on Ω.

An Application

The symmetric group Sym_n is particular for having elements of large order for relatively small degree. For instance, Sym_{28} has elements of order $2970 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, as $28 = 2+3+5+7+11$.

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This can't be said for other 'nearly simple' groups: Any finite simple G group has some natural representation, of degree $n(G)$, and **it has been observed experimentally that any $g \in G$ has at least one orbit of length $|g|$,** unless G is alternating or one of a few small exceptions. In particular, $|g| \leq n(G)$ for all $g \in G$.

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In fact, elements of order equal to $n(G)$ are usually quite special. (For instance, Singer cycles of the group, etc.)

What about an arbitrary representation of such a group?

Theorem (Siemons & A. Zalesskii, 2000 & 2002, CFSG):
Let G be a finite simple group. Assume that

(i) G is not an alternating group, and

(ii) G acts doubly transitively on some set Ω.

Let (G, Δ) be any non-trivial permutation representation of G.
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Sketch of Proof: (i) Show this is true for the natural doubly transitive representation; call this set Ω.

(ii) For the 'arbitrary' representation on Δ define the natural **standard homomorphism**

$$\varphi : F\Omega \ \rightarrow \ F\Delta$$

just as before

$$\varphi(\omega) := \sum_{g \in G_\omega} \delta^g.$$

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(iii) The exceptions occur exactly when we have a factorization of G in the form $G = G_\omega \cdot G_\delta$. Thankfully there is a known and short list of such factorisation for finite simple groups.

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• Beyond the immediate question of the FC other ideas are to be explored: (i) Association Schemes and Eigenvalue Distributions related to such representations; and (ii) Explicit Formulae for Character Multiplicities, in at least some cases.

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• Beyond the immediate question of the FC other ideas are to be explored: (i) Association Schemes and Eigenvalue Distributions related to such representations; and (ii) Explicit Formulae for Character Multiplicities, in at least some cases.

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Thank you!