q, t-Fuss-Catalan numbers for reflection groups

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- Let α be a nonzero vector in some real vector space V. We define s_α to be the *reflection in* V sending α to its negative while fixing pointwise the hyperplane H_α orthogonal to α.
- A *(finite) reflection group* W is a finite group generated by reflections.
- A *root system* Φ is a finite set of nonzero vectors in V satisfying the conditions:
  i. $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$,
  ii. the group $W_\Phi := \langle s_\alpha : \alpha \in \Phi \rangle$ permutes Φ among itself, i.e.

$$s_\alpha(\beta) \in \Phi$$

  for all $\alpha, \beta \in \Phi$.

Simple roots, positive roots

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Let Φ be a root system.

- A *simple system in* Φ is a subset $\Delta \subseteq \Phi$, such that
  - Δ forms a vector space basis of the $\mathbb{R}$-span of Φ in V and
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Remark

Any two simple (resp. positive) systems in Φ are conjugate under W_Φ.

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- Every finite reflection group W is equal to W_Φ for some root system Φ.
- The following root systems determine (up to isomorphisms) the irreducible finite reflection groups:

$$A_n, B_n, D_n \text{ and}$$
$$I_2(m), H_3, H_4, F_4, E_6, E_7, E_8.$$

Example: Type A_2

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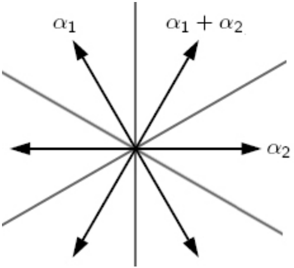
Let

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

Setting

$$\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3,$$

we can realize type A_2 as follows:



$$\begin{aligned} \Delta &= \{\alpha_1, \alpha_2\} \\ \Phi^+ &= \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \\ \Phi &= \Phi^+ \cup -\Phi^+ \\ W_\Phi &= \{\mathrm{id}, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\} \\ &\cong \mathcal{S}_3 \end{aligned}$$

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Alternating polynomials of type A

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Let Φ be the root system of type A_{n-1}. Then W_Φ is the symmetric group $\mathcal{S}_n$ and we define a *diagonal action* on

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$$

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$$\sigma(x_i) = x_{\sigma(i)}, \sigma(y_i) = y_{\sigma(i)} \text{ for } \sigma \in \mathcal{S}_n.$$

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$$\left[n = 2 : \begin{array}{ll} x_1y_1 - x_2y_2 & \text{alternating,} \\ x_1y_1 + x_2y_2 & \textbf{not} \text{ alternating} \end{array} \right]$$

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For Φ being the root system of type B_n, W_Φ is the group of *signed permutations* and the diagonal action is given by

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We call a polynomial f *alternating of type* Φ, if

$$\omega(f) = \det(\omega) f \text{ for all } \omega \in W_\Phi.$$

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The W_Φ-module $M_\Phi^{(m)}$ given by

$$M_\Phi^{(m)} := I^m / \langle \mathbf{x}, \mathbf{y} \rangle I^m$$

is the *minimal generating space* of I^m. It is naturally bigraded by degree in $\mathbf{x}$ and degree in $\mathbf{y}$, $M_\Phi^{(m)} = \oplus_{i,j \geq 0} M_{ij}$.

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Remark

As a vector space, $M_\Phi^{(m)}$ is isomorphic to the vector space with basis in one-to-one correspondence with a homogeneous minimal generating set of I^m.

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Definition

We define *q, t-Fuss-Catalan numbers* as the bigraded Hilbert series of $M_\Phi^{(m)}$, this is the generating function of the dimensions of its bigraded components:

$$\mathrm{Cat}_\Phi^{(m)}(q,t) := \mathcal{H}(M_\Phi^{(m)}; q, t) = \sum_{i,j \geq 0} \dim(M_{ij}) q^i t^j.$$

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- $\mathrm{Cat}_\Phi^{(m)}(q,t)$ is a symmetric polynomial in q and t.

The well-studied type A

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- Later it was shown to be equal to the Hilbert series of some cohomology module in the theory of *Hilbert schemes of points in the plane* (Haiman).
- It has a conjectured combinatorial interpretation in terms of two statistics on partitions that fit inside the partition $((n-1)m, \ldots, 2m, m)$:

$$\mathrm{Cat}^{(m)}_{A_{n-1}}(q,t) = \sum q^{\mathrm{area}(\lambda)} t^{\mathrm{dinv}(\lambda)}$$

  - proved for $m = 1$ (Garsia, Haglund) and for $t = 1$ (Haiman).

Example: $\mathrm{Cat}^{(1)}_{A_2}(q,t)$

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Let λ be a partition that fits inside the partition $(2,1)$. Then

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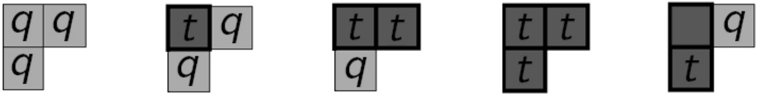
So, let us compute $\mathrm{Cat}^{(1)}_{A_2}(q,t) = \sum_\lambda q^{\mathrm{area}(\lambda)} t^{\mathrm{dinv}(\lambda)}$:

$\lambda = \emptyset$ $\qquad$ $\lambda = (1)$ $\qquad$ $\lambda = (2)$ $\qquad$ $\lambda = (2,1)$ $\qquad$ $\lambda = (1,1)$



$$\mathrm{Cat}^{(1)}_{A_2}(q,t) = q^3 + q^2 t + q t^2 + t^3 + qt$$

A conjectured formula for the dimension of $M_\Phi^{(m)}$

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Conjecture

$$\mathrm{Cat}_\Phi^{(m)}(1,1) = \prod_{i=1}^{l} \frac{d_i + mh}{d_i},$$

where l is the rank of Φ, h the Coxeter number and $d_1, \ldots, d_l$ the degrees of the *fundamental invariant of* Φ.

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- These numbers, called *Fuss-Catalan numbers*, count many combinatorial objects, for example *k-divisible non-crossing partitions* (Reiner, Bessis).

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A_{n-1}	B_n	D_n
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n}{n}$	$\binom{2n}{n} - \binom{2(n-1)}{n-1}$

$I_2(m)$	H_3	H_4	F_4	E_6	E_7	E_8
$m+2$	32	280	105	833	4160	25080

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The extended Shi arrangement

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Let Φ be crystallographic (A_{n-1}, B_n, D_n, $I_2(6)$, F_4, E_6, E_7, E_8).

We define the *extended Shi arrangement* $\mathrm{Shi}^{(m)}(\Phi)$ as

$$\bigcup_{\alpha \in \Phi^+, 0 \leq k \leq m} H_\alpha^k \subseteq V,$$

where $H_\alpha^k = \{x \in V : (x, \alpha) = k\}$.

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Remark

For $m = 0$, the extended Shi arrangement reduces to the Coxeter arrangement associated to Φ.

Example: $\mathrm{Shi}^{(2)}(A_2)$

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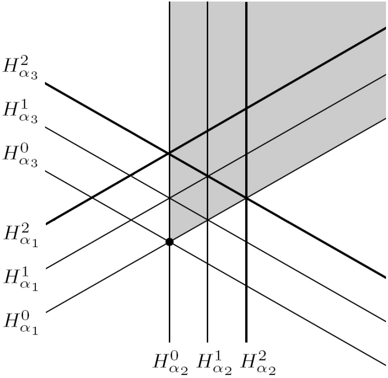

Number of regions of the extended Shi arrangement

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Theorem (C.A. Athanasiadis)

The number of regions into which the fundamental chamber of the Coxeter arrangement is dissected by the hyperplanes of $\mathrm{Shi}^{(m)}(\Phi)$ *equals*

$$\prod_{i=1}^{l} \frac{d_i + mh}{d_i},$$

where l *is the rank of* Φ*,* h *the Coxeter number and* $d_1, \ldots, d_l$ *the degrees of the fundamental invariant of* Φ*.*

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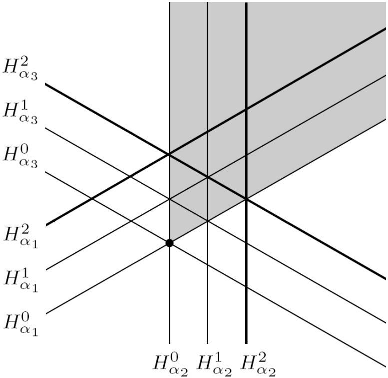

$$\prod_{i=1}^{l} \frac{d_i + mh}{d_i} = \frac{(d_1 + 2\cdot 3)(d_2 + 2\cdot 3)}{d_1 d_2} = \frac{(2+6)(3+6)}{6} = 12$$

The *coheight* of a region in the extended Shi arrangement

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Let R^∞ be the region given by $(x, \alpha) > m$ for all $\alpha \in \Phi^+$ and for any region R let $\mathrm{coh}(R)$ be the number of hyperplanes separating R from R^∞.

The *coheight* of a region in the extended Shi arrangement

Let R^∞ be the region given by $(x,\alpha) > m$ for all $\alpha \in \Phi^+$ and for any region R let $\mathrm{coh}(R)$ be the number of hyperplanes separating R from R^∞.

We combinatorially define *q-Fuss-Catalan numbers* by

$$\mathrm{Cat}_\Phi^{(m)}(q) := \sum q^{\mathrm{coh}(R)},$$

where the sum ranges over all regions of $\mathrm{Shi}^{(m)}(\Phi)$ which lie in the fundamental chamber of the Coxeter arrangement.

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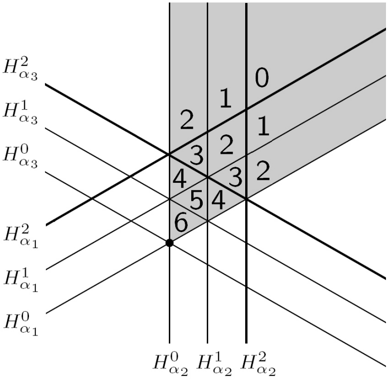

$$\mathrm{Cat}^{(2)}_{A_2}(q) = \sum q^{\mathrm{coh}(R)} = 1 + 2q + 3q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

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- also in type A, they satisfy the generating function identity

$$\sum_{n\geq 0} \frac{x^n q^{-(m+1)\binom{n}{2}}}{(-x;q^{-1})_{(m+1)n+1}} \mathrm{Cat}_{A_{n-1}}^{(m)}(q) = 1,$$

where $(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a)$.

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where $(a;q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a)$.

- For type B and $m=1$, they satisfy the generating function identity

$$\sum_{n\geq 0} \frac{x^n q^{-n(n-1)}(1-qx)}{(-x;q^{-1})_{2n+1}} \mathrm{Cat}_{B_n}^{(1)}(q) = 1.$$

The specialization $t = 1$ in $\mathrm{Cat}_{\Phi}^{(m)}(q, t)$

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- The conjecture is known to be true for type A,
- was validated by computations for
  - types B and D with $n \leq 4, m \leq 3$,
  - types $I_2(6)$ and F_4 with $m \leq 3$.

A remark on non-crystallographic root systems

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For crystallographic types, Φ^+ can be considered as a poset, the *root poset* of type Φ. It is defined by

$$\alpha \leq \beta :\Leftrightarrow \beta - \alpha \text{ is a positive linear combination of simple roots.}$$

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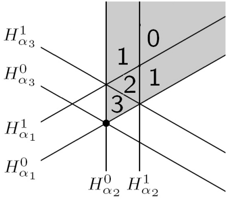
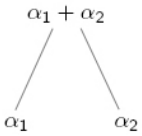


Type $A_2, m = 1$:

$$\leftrightsquigarrow 1 + 2q + q^2 + q^3 \leftrightsquigarrow$$



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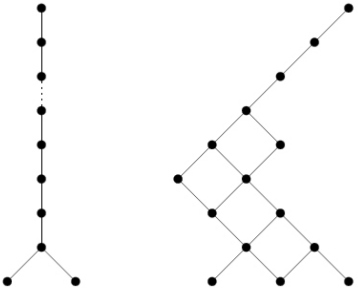
D. Armstrong suggested the following root posets for types $I_2(k)$ for any $k \geq 3$ and for type H_3:



Computing $\mathrm{Cat}_{\Phi}^{(1)}(q,t)$ gives:

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The specialization $t = q^{-1}$ in $\mathrm{Cat}_{\Phi}^{(m)}(q, t)$

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Conjecture

$$q^{mN}\,\mathrm{Cat}_\Phi^{(m)}(q,q^{-1}) = \prod_{i=1}^{l}\frac{[d_i+mh]_q}{[d_i]_q},$$

where N is the number of positive roots.

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Some facts about rational Cherednik algebras

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The *rational Cherednik algebra*

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- the polynomial rings $\mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*]$ and
- the group algebra $\mathbb{C}W$

are subalgebras of H_c.

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For $c = \frac{mh+1}{h}$ there exists a unique simple H_c-module $L(\mathrm{triv})$. It has following properties:

- $L(\mathrm{triv})$ is finite dimensional and graded ($\mathbf{e}$ respects this grading),
- the Hilbert series of its trivial component is given by

$$\mathcal{H}(\mathbf{e}L(\mathrm{triv}), q) = q^{-mN} \prod_{i=1}^{l} \frac{[d_i + mh]_q}{[d_i]_q}$$

The conjectured connection to $M_{\Phi}^{(m)}$

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Let $M_\Phi^{(m)}$ be graded by degree in $\mathbf{x}$ minus degree in $\mathbf{y}$. Then

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- The conjecture is known to be true in type A and
- would imply the conjectures about the specializations $t = q = 1$ and about the specialization $t = q^{-1}$.

H_c and the coinvariant ring

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For crystallographic reflection groups and $m = 1$, it is known that

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Conjecture

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- In these cases, the conjecture would imply the conjecture presented on the previous slide.

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- Think about the whole situation for well-generated complex reflection groups - the conjectures seem to be true also for those...