A characteristic free presentation of the ring of multisymmetric functions.

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- R is a commutative ring and let $n, m > 0$
- $A_R(n, m) = R[x_i(j)]$ with $i = 1, \ldots, m\,; j = 1, \ldots, n$
- The symmetric group on n letters S_n acts on $A_R(n, m)$ by means of $\sigma(x_i(j)) = x_i(\sigma(j))$
- $A_R(n, m)^{S_n}$ the rings of invariants for this action

- if $m = 1$, then $A_R(n, 1) \cong R[x_1, x_2, \ldots, x_n]$, and $R[x_1, x_2, \ldots, x_n]^{S_n}$ is freely generated by the elementary symmetric functions $e_1, \ldots, e_n$ given by the equality

$$1 + \sum_{k=1}^{n} t^k e_k := \prod_{i=1}^{n} (1 + tx_i)$$

t is a commuting independent variable.

$$e_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

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Unless otherwise stated, we now assume that $m > 1$.
We first obtain generators of the ring $A_R(n, m)^{S_n}$

- $A_R(m) = R[y_1, \ldots, y_m]$ and $f = f(y_1, \ldots, y_m) \in A_R(m)$
- $f(j) = f(x_1(j), \ldots, x_m(j))\ 1 \leq j \leq n$
- $f(j) \in A_R(n, m)\ \forall\ 1 \leq j \leq n$

- $\sigma(f(j)) = f(\sigma(j)),\ \forall \sigma \in S_n,\ j = 1, \ldots, n$
- $e_k(f) = e_k(f(1), f(2), \ldots, f(n))$ i.e.

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- One may think about the y_i as diagonal matrices in the following sense: let $M_n(A_R(n,m))$ be the full ring of $n \times n$ matrices with coefficients in $A_R(n,m)$. Then there is an embedding

$$\rho_n : A_R(m) \hookrightarrow M_n(A_R(n,m))$$

given by

$$\rho_n(y_i) := \begin{pmatrix} x_i(1) & 0 & \dots & 0 \\ 0 & x_i(2) & \dots & 0 \\ 0 & 0 & \dots & x_i(n) \end{pmatrix} \text{ for } i = 1, \dots, m.$$

so that

$$1 + \sum_{k=1}^{n} t^k e_k(f) = \prod_{j=1}^{n} (1 + t\rho_n(f)_{jj}) = det(1 + t\rho_n(f))$$

where $det(-)$ is the usual determinant of $n \times n$ matrices.

- Let $\mathcal{M}_m$ be the set of monomials in $A_R(m)$. For $\mu \in \mathcal{M}_m^+$ let $\partial_i(\mu)$ denote the degree of μ in y_i, for all $i = 1, \ldots, m$. We set

$$\partial(\mu) := (\partial_1(\mu), \ldots, \partial_m(\mu))$$

for its multidegree. The total degree of μ is $\sum_i \partial_i(\mu)$.
- Let $\mathcal{M}_m^+$ be the set of monomials of positive degree. A monomial $\mu \in \mathcal{M}_m^+$ is called *primitive* if it is not a power of another one. We denote by $\mathfrak{M}_m^+$ the set of primitive monomials.

- We define an S_n invariant multidegree on $A_R(n, m)$ by setting $\partial(x_i(j)) = \partial(y_i) \in \mathbb{N}^m$ for all $1 \leq j \leq n$ and $1 \leq i \leq m$.
- If $f \in A_R(m)$ is homogeneous of total degree l, then $e_k(f)$ has total degree kl (for all k and n).

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- If $f \in A_R(m)$ is homogeneous of total degree l, then $e_k(f)$ has total degree kl (for all k and n).

Theorem

The ring of multisymmetric functions $A_R(n,m)^{S_n}$ is generated by the $e_k(\mu)$, where $\mu \in \mathfrak{M}_m^+$, $k = 1, \ldots n$ and the total degree of $e_k(\mu)$ is less or equal than $n(m-1)$. If $n = p^s$ is a power of a prime and $R = \mathbb{Z}$ or $p \cdot 1_R = 0$, then at least one generator has degree equal to $m(n-1)$. If $R \supset \mathbb{Q}$ then $A_R(n,m)^{S_n}$ is generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$ and the degree of μ is less or equal than n.

- The action of S_n on $A_R(n,1) \cong R[x_1, x_2, \ldots, x_n]$ preserves the usual degree. We denote by $\Lambda^k_{R,n}$ the R-submodule of invariants of degree k.
- Let $q_n : R[x_1, x_2, \ldots, x_n] \to R[x_1, x_2, \ldots, x_{n-1}]$ be given by $x_n \mapsto 0$ and $x_i \mapsto x_i$, for $i = 1, \ldots, n-1$.
- This map sends $\Lambda^k_{n,R}$ to $\Lambda^k_{n-1,R}$ and it is easy to see that $\Lambda^k_{n,R} \cong \Lambda^k_{k,R}$ for all $n \geq k$.
- Denote by Λ^k_R the limit of the inverse system obtained in this way.
- The ring $\Lambda_R := \bigoplus_{k \geq 0} \Lambda^k_R$ is called the ring of *symmetric functions* (over R).

- It can be shown (Macdonald) that Λ_R is a polynomial ring, freely generated by the (limits of the) e_k, that are given by

$$1 + \sum_{k=1}^{\infty} t^k e_k := \prod_{i=1}^{\infty} (1 + t x_i)$$

- Furthermore the kernel of the natural projection $\pi_n : \Lambda_R \to \Lambda_{n,R}$ is generated by the e_{n+k}, where $k \geq 1$.

In a similar way we build a limit of multisymmetric functions.

- For any $a \in \mathbb{N}^m$ we set $A_R(n, m, a)$ for the linear span of the monomials of multidegree a. One has

$$A_R(n, m) = \bigoplus_{a \in \mathbb{N}^m} A_R(n, m, a)$$

- Let $\pi_n : A_R(n, m) \to A_R(n-1, m)$ be given by

$$\pi_n(x_i(j)) = \begin{cases} 0 & \text{if } j = n \\ x_i(j) & \text{if } j \leq n-1 \end{cases} \qquad \text{for all } i.$$

- Then $\forall a \in \mathbb{N}^m$

$$\pi_n(A_R(n, m, a)^{S_n}) = A_R(n-1, m, a)^{S_{n-1}}$$

- For any $a \in \mathbb{N}^m$ set

$$A_R(\infty, m, a) = \lim_{\leftarrow} A_R(n, m, a)^{S_n}$$

where the projective limit is taken with respect to n over the projective system $(A_R(n, m, a)^{S_n}, \pi_n)$.

- Set

$$A_R(\infty, m) = \bigoplus_{a \in \mathbb{N}^m} A_R(\infty, m, a)$$

- We set, by abuse of notation,

$$e_k(f) = \lim_{\leftarrow} e_k(f) \in A_R(\infty, m)$$

with $k \in \mathbb{N}$ and $f \in A(m)^+$, the augmentation ideal, i.e.

$$1 + \sum_{k=1}^{\infty} t^k e_k(f) := \prod_{j=1}^{\infty} (1 + tf(j))$$

- Then e_k is a homogeneous polynomial of degree k.
- If $f = \sum_{\mu \in \mathcal{M}_m^+} \lambda_\mu \mu$, we set

$$e_k(f) := \sum_{\alpha} \lambda^\alpha e_\alpha$$

where $\alpha := (\alpha_\mu)_{\mu \in \mathcal{M}_m^+}$ is such that $\alpha_\mu \in \mathbb{N}$, $\sum_{\mu \in \mathcal{M}_m^+} \alpha_\mu \leq k$ and $\lambda^\alpha := \prod_{\mu \in \mathcal{M}_m^+} \lambda^{\alpha_\mu}$.

Theorem

- *The ring $A_R(\infty, m)$ is a polynomial ring, freely generated by the (limits of) the $e_k(\mu)$, where $\mu \in \mathfrak{M}_m^+$ and $k \in \mathbb{N}$.*
- *The kernel of the natural projection*

$$A_R(\infty, m) \to A_R(n, m)^{S_n}$$

is generated as R-module by the coefficients e_α of the elements

$$e_{n+k}(f), \ \ where \ \ k \geq 1 \ \ and \ \ f \in A_R(m)^+.$$

Theorem

- *If R is an infinite field then the kernel of the natural projection is generated as an ideal by the elements*

$$e_{n+k}(f), \;\; \textit{where} \;\; k \geq 1 \;\; \textit{and} \;\; f \in A_R(m)^+.$$

- *If $R \supset \mathbb{Q}$ then $A_R(\infty, m)$ is freely generated by the $e_1(\mu)$, where $\mu \in \mathcal{M}_m^+$ and the kernel of the natural projection is generated as an ideal by the $e_{n+1}(f)$, where $f \in A_R(m)^+$.*