Conference on
Diagonally symmetric polynomials and applications

C.I.E.M., Castro–Urdiales, 15–19 october 2007

Abstracts

Talks by Invited speakers

François Bergeron
(Université du Québec à Montréal)
Reflection Groups and Diagonal Invariant Polynomials (2 talks)

Riccardo Biagioli
(Université de Lyon I)
Compact diagrams for signed permutation groups

Let $R := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the ring of polynomials in two set of variables. The module of coinvariants $R_{B_n \times B_n}$ is the quotient of R by the ideal $\mathcal{I}_{B_n \times B_n}$ generated by the constant term free invariant polynomials for the action of two copies of the hyperoctahedral group B_n. The set $R_{B_n \times B_n}$ is a bigraded B_n-module, where the group B_n acts in a diagonal way on the variables.

In this talk we will give an explicit description of the trivial and sign component of the B_n-representation $R_{B_n \times B_n}$. Two new classes of combinatorial objects, compact e and o-diagrams, will be introduced and used to give an explicit basis for the trivial component, and to compute the Hilbert series of the sign component, respectively.

The same setting will be generalized to the case of Weyl groups of type D. Some open problems will be discussed.

Ira Gessel
(Brandeis University)
Enumerative applications of MacMahon's symmetric functions of several systems of quantities.

P. A. MacMahon studied diagonally symmetric functions in several sets of variables and applied them to enumerative problems. I will talk about MacMahon's symmetric functions from a more modern point of view and describe several of their applications. I will also talk about the ways that they are similar to and different from the familiar symmetric functions of one set of variables.

Marni Mishna
(Simon Fraser University.)

Closure Properties of Holonomic D-Modules Useful in Combinatorics (2 talks)

The presence and importance of D-finite functions in combinatorics is on the rise. By definition, D-finite functions satisfy linear differential equations with polynomial coefficients, and they correspond to holonomic D-modules; indeed, they are often called holonomic functions. In this talk we will explore the D-module side of the story, and consider how we can, in certain cases, describe an algorithm to compute the scalar product of symmetric functions, and the consequences of this computation in enumerative, and algebraic combinatorics. We will also consider recent progress and applications of the D-module point of view to efficiency improvements in recurrence manipulation, and in other coefficient extraction scenarios.

Francesco Vaccarino
(Politecnico di Torino)

A characteristic free presentation of the ring of multisymetric functions

We will give a characteristic free presentation of the ring of multisymmetric functions. The special case of an infinite field will be treated in great details.

Multisymmetric functions and invariants of matrices.

We will show how the multisymmetric functions can be seen as invariants of matrices, giving a full generalization of a classical result due to H.Weyl.

Contributed talks

Ron Adin
(Bar–Ilan University)

Group basis, flag major index, and invariant polynomials.

The Hilbert series of the algebra of diagonally invariant polynomials (under the natural action of the symmetric group) may be elegantly described in terms of the major index statistic, as first shown by Garsia and Gessel. We shall describe several extensions of this result to other groups, using suitably extended versions of the major index, and relate it to the concept of group basis.

Based on joint works with Oshrit Ovrutzky, Yuval Roichman and Robert Shwartz.

Emmanuel Briand
(Universidad de Sevilla)

Diagonally symmetric polynomials in computational algebra

I will explain why diagonally symmetric polynomials were introduced at the end of the XIX$^{\text{th}}$ century by eminent mathematicians (*e.g.* Schläfli, Cayley, Brill) to study problems that nowadays belong to the field of computational algebra: solving systems of polynomial equations with finitely many solutions, and determining when an algebraic form is totally decomposable (*i.e.* factorizes as a product of linear forms). Modern developments of their work will be presented. I will also evoke the problem of computing the algebraic relations between the elementary diagonally symmetric polynomials.

Foulkes' conjecture and diagonally symmetric polynomials

I will explain how the Foulkes conjecture (see the abstract of J. Siemons talk) is related to the geometry of the subvariety of decomposable forms (products of linear forms), the algebra of homogenous diagonally symmetric polynomials, and the invariants of $m \times n$ matrices under $S_m \times S_n$ (row permutations and column permutations).

Yona Cherniavsky
(Technion)

Permutation representations on invertible matrices.
(joint work with Eli Bagno)

We discuss permutation representations which are obtained by the natural action of $S_n \times S_n$ and S_n itself as its diagonal subgroup on some special sets of invertible $(0,1)$–matrices, defined by simple combinatorial attributes. We decompose these representations into irreducibles. The multiplicities involved have a nice combinatorial interpretation. We also generalize known results on asymptotic behavior of the conjugacy representation of S_n. This work is a part of my Ph.D. thesis supervised by Professor Yuval Roichman, Bar Ilan University , Israel.

Peter Fleischmann
(Institute for Mathematics, Statistics and Actuarial Science,
University of Kent at Canterbury)

On modular invariants of finite groups.

Let F be a field and G a finite group, acting on the polynomial ring $A := F[x_1, \ldots, x_d]$ by graded F - algebra automorphisms. The ring of invariants $A^G := \{f \in A \mid g(f) = f\}$ is the main object of study in Invariant Theory. The theory is very well developed in the "classical case", where the characteristic of F is zero, but far less so in the case of positive characteristic p, in particular the "modular case", where p divides the group order $|G|$.

In that case there are open questions about the constructive complexity of A^G, measured by degree bounds for generators, and about the structural complexity, measured by the co-depth (=Krull-dimension - maximal length of regular sequence) of A^G.

In my talk I will report on some recent results dealing with both types of questions.

They include the recent solution of the degree bound problem for modular invariant rings of arbitrary finite-dimensional $\mathbb{Z}/p\mathbb{Z}$-representations. We also present a new method to construct A^G, which was developed by Kemper, Woodcock and the author. It is based on ideas from number theory, and provides a way how to make use of diagonal invariants of the symmetric group and the Noether-homomorphism in the modular situation.

Avital Frumkin
(School of Mathematical Sciences, Tel Aviv University)

Asymptotics of Kronecker coefficients for Diagrams in a given k, ℓ–hook

Ming-chang Kang
(National Taiwan University)

Rationality of $GL(2,3)$: the characteristic two case.

Let K be any field and G be a finite group. Let G act on the rational function field $K(x_g : g \in G)$ by K-automorphisms and $h \cdot x_g = x_{hg}$ for any $g, h \in G$. Define $K(G) = K(x_g : g \in G)^G$ to be the fixed field of $K(x_g : g \in G)$ under the action of G. Noether's problem asks under what situations the field $K(G)$ is rational (= purely transcendental) over K.

Noether's problem is related to the inverse Galois problem which asks whether there is a Galois extension field L over K with $Gal(L/K) \simeq G$, provided that K is a prescribed algebraic number field and G is a prescribed finite group. In fact, if K is an algebraic number field and $K(G)$ is rational over K, then $K(G)$ is retract rational (equivalently, there exists a generic Galois G-extension over K), which will guarantee the existence of the inverse Galois problem for K and G.

Noether's problem was proved in the affirmative by Fischer (1916) when G is an abelian group and K contains enough roots of unity. Furtwängler was able to show that $\mathbb{Q}(G)$ is rational over $\mathbb{Q}$ if G is a Frobenius group of order pd where $p = 3, 5, 7$ or 11, and d is a divisor of $p - 1$. The first counter-example to Noether's problem was found by Swan (1969): $\mathbb{Q}(G)$ is not rational if G is a cyclic group of order $47, 113$ or 233, etc. Noether's problem for abelian groups was solved by H.

W. Lenstra, Jr.(1974). The knowledge about Noether's problem for non-ableian groups is rather scarce.

After a brief survey of Noether's problem, we will focus on two specific examples. The first one is the A_4-action on $K(x_1, x_2, x_3, x_4)$, whose rationality problem was solved by Burnside (1907) with a set of generators. Unfortunately Burnside's formula is not correct. We will give a correct formula and explain the reason how the mistake arose. The second example is the rationality problem for $GL(2, 3)$. Note that the group $GL(2, 3)$ is isomorphic to $\widetilde{S}_4$, a double cover of S_4 and the rationality of $K(\widetilde{S}_4)$ was proved by B. Plans recently; we will give a new proof of Plans's Theorem. We would like to remark that, for another double cover of S_4, $\widehat{S}_4$, Serre is able to show that $\mathbb{Q}(\widehat{S}_4)$ is not stably rational over $\mathbb{Q}$.

Thomas McKay
(University of East Anglia)

On a conjecture of Foulkes (II)

This talk follows on from that of Johannes Siemons. Let H^λ denote the F-vectorspace whose basis is the set of λ-partitions of the set $\{1, ..., n\}$. Independantly Siemons and Wagner and Stanley have conjectured that a specific map injects H^λ into $H^{\lambda'}$. In this talk we report on recent work on this conjecture.

Yuval Roichman
(Bar–Ilan University)

The combinatorics of diagonal harmonics of hook shape.

The solution of Macdonald's positivity conjectures involves modules of diagonal harmonics, which were introduced by Garsia and Haiman in the early nineties. These modules are spanned by all partial derivatives of generalized Vandermonde determinants. In this talk we will describe the combinatorial structure of the modules associated with hook shapes.

In particular, several monomial bases are presented and an explicit formula for the diagonal S_n-action is given. A combinatorial decomposition rules, which extend a classical result of Lusztig and Stanley, are deduced. Variants of descent and inversion statistics on permutations and tableaux play a crucial role in these descriptions.

Mercedes Rosas
(Universidad de Sevilla)

Symmetric Functions in Noncommutative variables, and MacMahon symmetric functions

We start by introducing the algebra of symmetric functions in noncommutative variables. Then, we show how to use them to manipulate MacMahon symmetric functions. The partition lattice will play a central role in this combinatorial framework.

Johannes Siemons
(University of East Anglia)

On a conjecture of Foulkes (I)

For the integers a and b let $P(a^b)$ be all partitions of the set $\{1, \ldots, ab\}$ into parts of size a and let $\mathbb{C}P(a^b)$ be the corresponding permutation module for the symmetric group Sym_{ab}. A conjecture of Foulkes says that $\mathbb{C}P(a^b)$ is isomorphic to a submodule of $\mathbb{C}P(b^a)$ for all $a \geq b$. I will discuss some proofs of the conjecture for small values of b.

We turn to a more general question. Suppose that the group G acts as a permutation group on two sets Ω and Δ. What information can be derived from the existence of an injective $\mathbb{C}G$- homomorphisms $\mathbb{C}\Omega \to \mathbb{C}\Delta$?

Christian Stump
(University of Vienna)

q,t–Catalan numbers for reflection groups.

In type A, the q,t–Fuß-Catalan numbers $\mathrm{Cat}_n^{(m)}(q,t)$ can be defined as a bigraded Hilbert series of a module associated to the symmetric group Sn. We generalize this construction to arbitrary (finite) reflection groups and exhibit some nice conjectured algebraic and combinatorial properties of these polynomials in q and t. Finally, we present an idea of how these polynomials could be related to some graded Hilbert series of modules arising in the context of rational Cherednik algebras. This talk is based on work in progress.