Computational Aspects of Invariants of Multipartite Quantum Systems

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Main Problem

Characterize the non-local properties of quantum states.

Various approaches

- entanglement measures:
  (real) functions on the state space, *e. g.* distance to product/separable states

- local equivalence:
  Given two quantum states

$$|\psi\rangle \text{ and } |\phi\rangle \qquad (\rho \text{ and } \rho')$$

on n particles (qudits), is there a local *unitary*[a] transformation $U = U_1 \otimes U_2 \otimes \ldots \otimes U_n$ with

$$U|\psi\rangle = |\phi\rangle \qquad (U\rho U^{-1} = \rho')?$$

[a] We do not consider SLOCC here.

Our Approach

Use the polynomial invariants of the groups

- $SU(d_1) \otimes \ldots \otimes SU(d_n)$
- $U(d_1) \otimes \ldots \otimes U(d_n)$

operating on

- pure states $|\psi\rangle$
- mixed states ρ

to describe multi-particle entanglement.

This gives a *complete* description:

Theorem:
The orbits of a compact linear group acting in a *real* vector space are separated by the (polynomial) invariants.

(A. L. Onishchik, *Lie groups and algebraic groups*, Springer, 1990, Ch. 3, §4)

Operation of $GL(d, \mathbb{K})$

pure quantum states:

linear operation on polynomials $f \in \mathbb{K}[x_1, \ldots, x_d] =: \mathbb{K}[\boldsymbol{x}]$

$$f(\boldsymbol{x})^g := f(\boldsymbol{x}^g) \qquad \text{where } \boldsymbol{x}^g = (x_1, \ldots, x_d) \cdot g \text{ and } g \in GL(d, \mathbb{K})$$

mixed quantum states:

operation on polynomials $f \in \mathbb{K}[x_{11}, \ldots, x_{dd}] =: \mathbb{K}[X]$ via conjugation

$$f(X)^g := f(X^g) \quad \text{where}$$

$$X^g = g^{-1} \cdot \begin{pmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & \ddots & \vdots \\ x_{d1} & \cdots & x_{dd} \end{pmatrix} \cdot g$$

Polynomial Invariants

Properties of $\mathbb{K}[\boldsymbol{x}]^G := \{f(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}] | \forall g \in G\colon f(\boldsymbol{x})^g = f(\boldsymbol{x})\}$

- Homogeneous polynomials remain homogeneous
  $\Longrightarrow$ homogeneous generators.
- Any linear combination of invariants is an invariant.
- The product of invariants is an invariant.
- For reductive groups $\mathbb{K}[\boldsymbol{x}]^G$ is finitely generated.
- Some invariants are algebraically independent (primary invariants).
- The other invariants obey some polynomial relations.
- In special cases: the invariant ring can be decomposed as a free module (generated by the secondary invariants) over the primary invariants.

Reynolds Operator

finite groups

$$\begin{array}{rcl} R_G\colon \quad \mathbb{K}[\boldsymbol{x}] & \to & \mathbb{K}[\boldsymbol{x}]^G \\ f(\boldsymbol{x}) & \mapsto & \displaystyle\frac{1}{|G|}\sum_{g\in G} f(\boldsymbol{x})^g \end{array}$$

R_G is a linear projection operator

$\Rightarrow$ compute $R_G(\boldsymbol{m})$ for all monomials $\boldsymbol{m} \in \mathbb{K}[\boldsymbol{x}]$ of degree $k = 1, 2, \ldots$

compact groups

$$\begin{array}{rcl} R_G\colon \quad \mathbb{K}[\boldsymbol{x}] & \to & \mathbb{K}[\boldsymbol{x}]^G \\ f(\boldsymbol{x}) & \mapsto & \displaystyle\int_{g\in G} f(\boldsymbol{x})^g d\mu_G(g) \end{array}$$

where $\mu_G(g)$ is the normalized Haar measure of G

Problem: computing the integral is very difficult

Invariant Polynomials and Commuting Matrices

Every homogeneous polynomial $f(X) \in \mathbb{K}[x_{11}, \ldots, x_{dd}]$ of degree k can be expressed as

$$f_F(X) := \operatorname{tr}(F \cdot X^{\otimes k}) \quad \text{where } F \in \mathbb{K}^{kd \times kd}$$

(since $X^{\otimes k}$ contains all monomials of degree k).

Example ($n = 2$, $k = 2$):

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

$$X^{\otimes 2} = \begin{pmatrix} x_{11}^2 & x_{11}x_{12} & x_{12}x_{11} & x_{12}^2 \\ x_{11}x_{21} & x_{11}x_{22} & x_{12}x_{21} & x_{12}x_{22} \\ x_{21}x_{11} & x_{21}x_{12} & x_{22}x_{11} & x_{22}x_{12} \\ x_{21}^2 & x_{21}x_{22} & x_{22}x_{21} & x_{22}^2 \end{pmatrix}$$

Invariant Polynomials and Commuting Matrices

$$\begin{aligned} f_F(X)^g &= \operatorname{tr}(F \cdot (g^{-1} \cdot X \cdot g)^{\otimes k}) \\ &= \operatorname{tr}(F \cdot (g^{-1})^{\otimes k} \cdot X^{\otimes k} \cdot g^{\otimes k}) \\ &= \operatorname{tr}(g^{\otimes k} \cdot F \cdot (g^{-1})^{\otimes k} \cdot X^{\otimes k}) \\ &= \operatorname{tr}(F^{(g^{-1})^{\otimes k}} \cdot X^{\otimes k}) \end{aligned}$$

$$f_F(X)^g = f_F(X) \Longleftrightarrow f_F(X) = f_{F'}(X) \quad \text{and} \quad F' \cdot g^{\otimes k} = g^{\otimes k} \cdot F'$$

transformed question

Which matrices commute with each $g^{\otimes k}$ for $g \in G$?

R. Brauer (1937):

The algebra $\mathcal{A}_{d,k}$ of matrices that commute with each $U^{\otimes k}$ for $U \in U(d)$ is generated by a certain representation of S_k.

One Particle

- Hilbert space $\mathcal{H}$ of dimension d
- $G = U(d)$
- representation of S_k:
  S_k operates on a tensor product of k Hilbert spaces $\mathcal{H}_i$ of dimension d by permuting the spaces:

$$T_{d,k}(\pi) \cdot (\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_k) = \mathcal{H}_{\pi(1)} \otimes \ldots \otimes \mathcal{H}_{\pi(k)}$$

- "permuting k copies of $\mathcal{H}$"

N Particles

- Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$
- $G = U(d)^{\otimes N}$, $g = U_1 \otimes \ldots \otimes U_N$,
  $g^{\otimes k} = (U_1 \otimes \ldots \otimes U_N) \otimes \ldots \otimes (U_1 \otimes \ldots \otimes U_N)$
- N permutations $\pi_\nu \in S_k$
- representation of $(S_k)^N$:
  $\pi = (\pi_1, \ldots, \pi_N)$, π_ν permutes the copies of the ν^{th} particle:

$$T_{d,k}^{(N)}(\pi) \cdot \big((\mathcal{H}_{1,1} \otimes \ldots \otimes \mathcal{H}_{N,1}) \otimes \ldots \otimes (\mathcal{H}_{1,k} \otimes \ldots \otimes \mathcal{H}_{N,k})\big) =$$

$$(\mathcal{H}_{1,\pi_1(1)} \otimes \ldots \otimes \mathcal{H}_{N,\pi_N(1)}) \otimes \ldots \otimes (\mathcal{H}_{1,\pi_1(k)} \otimes \ldots \otimes \mathcal{H}_{N,\pi_N(k)})\big)$$

Computing Invariants

(see E. Rains, quant-ph/9704042[a]; Grassl et al., quant-ph/9712040[b])

Computing the homogeneous polynomial invariants of degree k for an N particle system with density operator ρ:

for each N tuple $\pi = (\pi_1, \ldots, \pi_N)$ of permutations $\pi_\nu \in S_k$ compute

$$f_{\pi_1,\ldots,\pi_N}(\rho_{ij}) := \operatorname{tr}\left(T_{d,k}^{(N)}(\pi) \cdot \rho^{\otimes k}\right)$$

- all homogeneous polynomial invariant of degree k
- in general, $(k!)^N$ invariants to compute
- not necessarily linearly independent, not even distinct
- it is sufficient to consider certain tuples of permutations

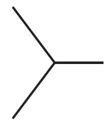
[a] IEEE Transactions on Information Theory, vol. 46, no. 1, pp. 54–59 (2000)
[b] Physical Review A 58, 1833–1839 (1998)

Invariant Tensors

- use local basis for the density matrix:

$$\rho = \frac{1}{4} I + \sum_{i=x,y,z} s_i \sigma_i \otimes I + \sum_{j=x,y,z} p_j I \otimes \sigma_j + \sum_{i,j=x,y,z} \beta_{ij} \sigma_i \otimes \sigma_j$$

- $SU(2) \otimes SU(2)$ acts as $SO(3) \times SO(3)$ on the coefficient vectors s, p and the coefficient matrix β
- contract copies of the coefficient tensors with tensors that are invariant under $SO(3)$ resp. $SO(3) \times SO(3)$

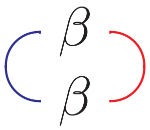
δ_{ij}	inner product	
ϵ_{ijk}	determinant	

- create all possible contractions modulo the relations of the tensors

for two qubits, there is only a finite number of such contractions

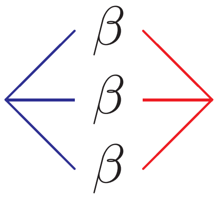
$\Longrightarrow$ complete set of invariants, resp. a set of generators for all invariants

Fundamental Invariants (I)

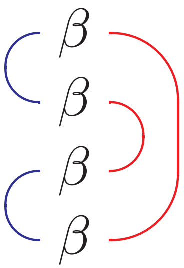
$\mathrm{Tr}(\beta\beta^t) =$ 

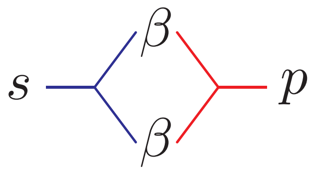
$s^t\, s = s$ — s

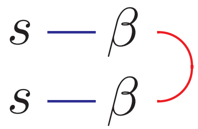
$p\, p^t = p$ — p

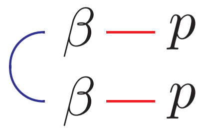
$\det\beta =$ 

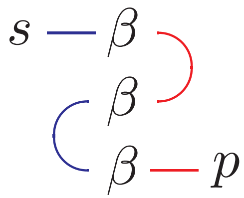
$s^t\, \beta\, p = s$ — β — p

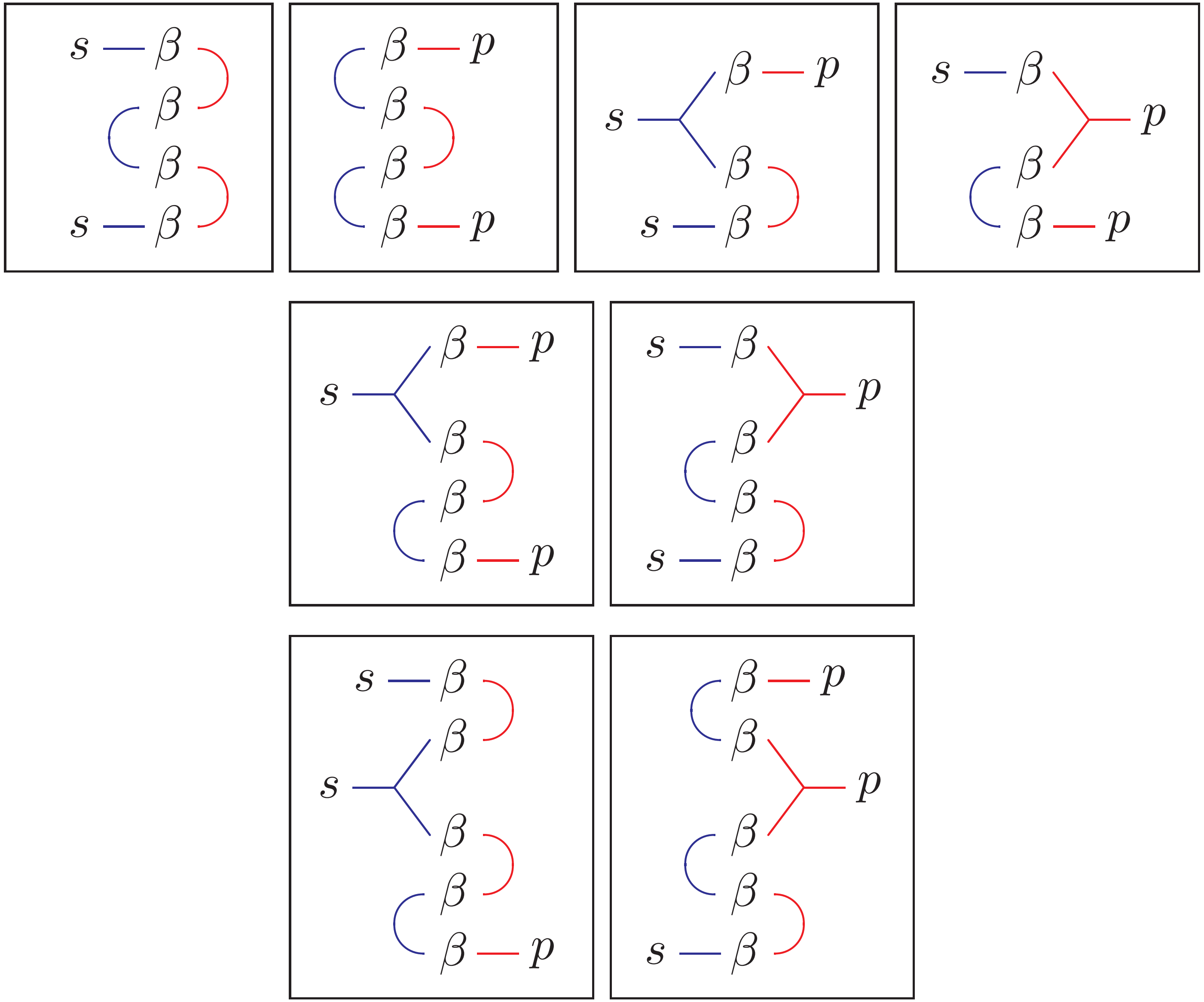


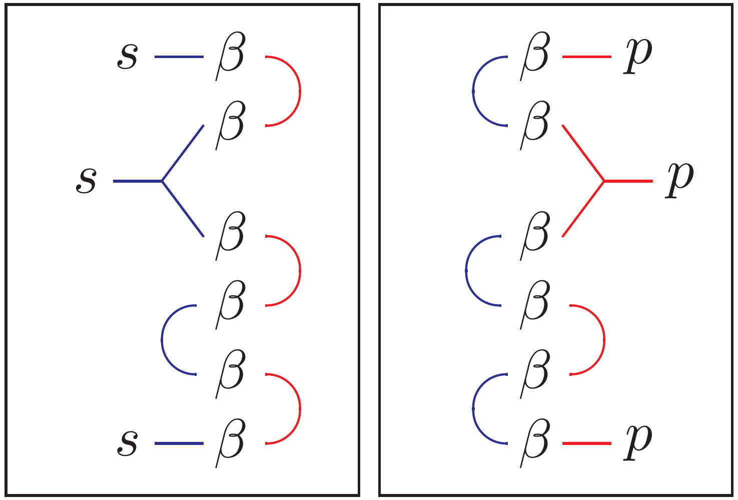












References

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Grassl, Entanglement and Invariant Theory, Quantum Computation and Information Seminar, UC Berkeley, 19.11.2002.

King, Welsh & Jarvis, The mixed two-qubit system and the structure of its ring of local invariants, J. Phys. A. 40, 10083–10108 (2007).

Hilbert Series

- encodes the vector space dimension d_k of the homogeneous invariants of degree k as a formal power series with non–negative integer coefficients:

$$M(z) := \sum_{k\geq 0} d_k z^k \in \mathbb{Z}[\![z]\!]$$

- a rational function (for finitely generated algebras)
- general formula (for linear operation)

$$M(z) = \int_{g\in G} \mathrm{d}\mu_G(g) \frac{1}{\det(id - z\cdot g)}$$

1. applies only to the case of linear operation
   $\Longrightarrow$ "linearize" the operation by conjugation via the adjoint representation
2. integral is very difficult to compute

Hilbert Series via Kronecker Coefficients

(see King *et al.*)

- the number of invariants d_m of degree m corresponds to the multiplicity of the trivial representation in the m-th symmetric power
- via branching rules for the restricted representation one obtains

$$d_m = \sum_{\lambda \vdash m; \ell(\lambda) \leq 4} \left(\sum_{\mu \vdash m; \ell(\mu) \leq 2} k^{\lambda}_{\mu\mu} \right)^2$$

$$= \sum_{\lambda \vdash m; \ell(\lambda) \leq 4} \left(\sum_{\mu, \nu \vdash m; \ell(\mu), \ell(\nu) \leq 2} (k^{\lambda}_{\mu\nu})^2 \right)$$

Example: Two Qubits

pure state

$$|\psi\rangle = x_{00}|00\rangle + x_{01}|01\rangle + x_{10}|10\rangle + x_{11}|11\rangle$$

Invariants

$$\mathrm{tr}(|\psi\rangle\langle\psi|) = x_{00}\overline{x}_{00} + x_{01}\overline{x}_{01} + x_{10}\overline{x}_{10} + x_{11}\overline{x}_{11}$$

$$\begin{aligned} \mathrm{tr}\left((\mathrm{tr}_i\,|\psi\rangle\langle\psi|)^2\right) &= x_{00}^2\overline{x}_{00}^2 + x_{01}^2\overline{x}_{01}^2 + x_{10}^2\overline{x}_{10}^2 + x_{11}^2\overline{x}_{11}^2 \\ &\quad +2x_{00}x_{01}\overline{x}_{00}\overline{x}_{01} + 2x_{00}x_{10}\overline{x}_{00}\overline{x}_{10} + 2x_{00}x_{11}\overline{x}_{01}\overline{x}_{10} \\ &\quad +2x_{01}x_{10}\overline{x}_{00}\overline{x}_{11} + 2x_{01}x_{11}\overline{x}_{01}\overline{x}_{11} + 2x_{10}x_{11}\overline{x}_{10}\overline{x}_{11} \end{aligned}$$

Problem

We have to introduce new variables which are the "complex conjugated variables".

Multivariate Hilbert Series

- operation on polynomials $f(\boldsymbol{x}, \overline{\boldsymbol{x}})$ in variables x_i and $\overline{x}_i$ with the representation $g \oplus \overline{g}$
- bi-degree

$$(\deg_{x_1,\ldots,x_d} f,\ \deg_{\overline{x}_1,\ldots,\overline{x}_d} f)$$

- invariant ring admits bi-graduation with Hilbert series

$$M(z, \overline{z}) := \sum_{k,\ell \geq 0} d_{k,\ell}\, z^k\, \overline{z}^\ell \in \mathbb{Z}[\![z, \overline{z}]\!]$$

- general formula (for linear operation)

$$M(z, \overline{z}) = \int_G \mathrm{d}\mu_G(g) \frac{1}{\det(id - z \cdot g)} \frac{1}{\det(id - \overline{z} \cdot \overline{g})}$$

Three Qubits: Ansatz for Series of $SU(2)^{\otimes 3}$

$$H_{SU}(\overline{z}, z) = \int_{U \in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z \cdot U)} \frac{1}{\det(id - \overline{z} \cdot U^t)}$$

$$= \frac{1}{(2\pi i)^3} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-v^2)(1-w^2)(1-x^2)}{\prod\limits_{a,b,c \in \{1,-1\}} (1 - z \cdot v^a w^b x^c)\,(1 - \overline{z} \cdot v^a w^b x^c)} \frac{\mathrm{d}v}{v} \frac{\mathrm{d}w}{w} \frac{\mathrm{d}x}{x}$$

($G = SU(2)^{\otimes 3}$, $U = U_1 \otimes U_2 \otimes U_3$, $\Gamma =$complex unit circle)

Computation of the integral using the theorem of residues

- symbolic computation of singularities and residues
- data type: factored rational functions implemented in MAGMA
  (back in 1997, Maple fails: "object too large")

Three Qubits: Series for $SU(2)^{\otimes 3}$ and $U(2)^{\otimes 3}$

$$H_{SU}(z,\overline{z}) = \frac{z^5\overline{z}^5 + z^3\overline{z}^3 + z^2\overline{z}^2 + 1}{(1-z\overline{z})(1-z^4)(1-\overline{z}^4)(1-z^2\overline{z}^2)^2(1-z\overline{z}^3)(1-z^3\overline{z})}$$

$$\begin{aligned} &= 1 + z\overline{z} + z^4 + z^3\overline{z} + 4z^2\overline{z}^2 + z\overline{z}^3 + \overline{z}^4 + z^5\overline{z} + z^4\overline{z}^2 + 5z^3\overline{z}^3 + z^2\overline{z}^4 + z\overline{z}^5 \\ &\quad + z^8 + z^7\overline{z} + 5z^6\overline{z}^2 + 5z^5\overline{z}^3 + 12z^4\overline{z}^4 + 5z^3\overline{z}^5 + 5z^2\overline{z}^6 + z\overline{z}^7 + \overline{z}^8 \\ &\quad + z^9\overline{z} + z^8\overline{z}^2 + 6z^7\overline{z}^3 + 6z^6\overline{z}^4 + 15z^5\overline{z}^5 + z\overline{z}^9 + z^2\overline{z}^8 + 6z^3\overline{z}^7 + 6z^4\overline{z}^6 \\ &\quad + z^{12} + z^{11}\overline{z} + 5z^{10}\overline{z}^2 + 6z^9\overline{z}^3 + 16z^8\overline{z}^4 + 16z^7\overline{z}^5 + 30z^6\overline{z}^6 \\ &\qquad + \overline{z}^{12} + z\overline{z}^{11} + 5z^2\overline{z}^{10} + 6z^3\overline{z}^9 + 16z^4\overline{z}^8 + 16z^5\overline{z}^7 \\ &\quad + \ldots \end{aligned}$$

$$\begin{aligned} H_U(z) &= \frac{z^{12}+1}{(1-z^2)(1-z^4)^3(1-z^6)(1-z^8)} \\ &= 1 + z^2 + 4z^4 + 5z^6 + 12z^8 + 15z^{10} + 30z^{12} + 37z^{14} + 65z^{16} + 80z^{18} \\ &\quad + 128z^{20} + 156z^{22} + 234z^{24} + 282z^{26} + 402z^{28} + 480z^{30} + \ldots \end{aligned}$$

Three Qubits: Invariant Ring of $SU(2)^{\otimes 3}$

Coefficient vector:

$$\boldsymbol{x} = \Big(\underbrace{x_{000}, x_{001}}_{00}, \underbrace{x_{010}, x_{011}}_{01}, \underbrace{x_{100}, x_{101}}_{10}, \underbrace{x_{110}, x_{111}}_{11}\Big)$$

Invariants of $I_4 \otimes SU(2)$:

brackets	$[i,j]$	$:=$	$x_{i0}x_{j1} - x_{i1}x_{j0}$	invariant of $SL(2) \supset SU(2)$
inner products	$\langle i,j\rangle$	$:=$	$x_{i0}\overline{x}_{j0} + x_{i1}\overline{x}_{j1}$	

Invariants of $U(1) \otimes SU(2) \otimes SU(2) \otimes SU(2)$:

correspond to permutations (π_1, π_2, π_3):

$$f_{\pi_1,\pi_2,\pi_3} = \sum_{i,j,\ldots} x_{i_1,i_2,i_3}\overline{x}_{\pi_1(i_1),\pi_2(i_2),\pi_3(i_3)} \cdot x_{j_1,j_2,j_3}\overline{x}_{\pi_1(j_1),\pi_2(j_2),\pi_3(j_3)} \cdot \ldots$$

Three Qubits: Invariant Ring of $SU(2)^{\otimes 3}$

Generators:

	bi-degree	permutations (π_1, π_2, π_3), brackets, inner products	#terms
f_1	$(1,1)$	(id, id, id)	8
f_2	$(2,2)$	$((1,2),(1,2),id)$	36
f_3	$(2,2)$	$((1,2),id,(1,2))$	36
s_1	$(4,0)$	$[1,2]^2 - 2[0,1][2,3] - 2[0,2][1,3] + [0,3]^2$	12
$\overline{s_1}$	$(0,4)$	$\overline{[1,2]}^2 - 2\overline{[0,1]}\,\overline{[2,3]} - 2\overline{[0,2]}\,\overline{[1,3]} + \overline{[0,3]}^2$	12
s_2	$(3,1)$	$[3,0]\langle 0,0\rangle - [3,0]\langle 3,3\rangle + [3,1]\langle 0,1\rangle + [3,2]\langle 0,2\rangle$ $+2[3,2]\langle 1,3\rangle - 2[1,0]\langle 2,0\rangle - [1,0]\langle 3,1\rangle - [2,0]\langle 3,2\rangle$ $-[2,1]\langle 0,0\rangle - [2,1]\langle 1,1\rangle + [2,1]\langle 2,2\rangle + [2,1]\langle 3,3\rangle$	40
$\overline{s_2}$	$(1,3)$		40
f_4	$(2,2)$	$(id,(1,2),(1,2))$	36
f_5	$(3,3)$	$((1,2),(2,3),(1,3))$	176
$f_4 f_5$	$(5,5)$		3760

Three Qubits: Invariant Ring of $U(2)^{\otimes 3}$

Generators of the invariant ring:

	degree	permutations (π_1, π_2, π_3)	#terms
f_1	2	(id, id, id)	8
f_2	4	$((1,2),(1,2),id)$	36
f_3	4	$((1,2),id,(1,2))$	36
f_4	4	$(id,(1,2),(1,2))$	36
f_5	6	$((1,2),(2,3),(1,3))$	176
f_6	8	$s_1\overline{s}_1$	144
f_7	12	$\overline{s}_1 s_2^2$	5988

f_1,. . . ,f_6 are algebraic independent; relation for f_7:

$$f_7^2 + c_1(f_1, \ldots, f_6) f_7 + c_0(f_1, \ldots, f_6) \qquad \text{where } c_0, c_1 \in \mathbb{Q}[f_1, \ldots, f_6]$$

completeness can be shown using the fact that there is only one algebraic relation

Four Qubits: Ansatz for Series of $SU(2)^{\otimes 4}$

$$H_{SU}(\overline{z}, z) = \int_{U \in G} \mathrm{d}\mu_G(U) \frac{1}{\det(id - z \cdot U)} \frac{1}{\det(id - \overline{z} \cdot U^t)}$$

$$= \alpha \oint_{\Gamma_u} \oint_{\Gamma_v} \oint_{\Gamma_w} \oint_{\Gamma_x} \frac{(1-u^2)(1-v^2)(1-w^2)(1-x^2)}{\prod\limits_{a,b,c,d \in \{1,-1\}} (1 - z \cdot u^a v^b w^c x^d)(1 - \overline{z} \cdot u^a v^b w^c x^d)} \frac{\mathrm{d}u}{u} \frac{\mathrm{d}v}{v} \frac{\mathrm{d}w}{w} \frac{\mathrm{d}x}{x}$$

Four Qubits: Hilbert Series of $SU(2)^{\otimes 4}$

$$H_{SU}(z,\overline{z}) = \big(z^{36}\overline{z}^{36} - z^{35}\overline{z}^{33} + 2z^{34}\overline{z}^{34} + 6z^{34}\overline{z}^{32} + 9z^{34}\overline{z}^{30} + 4z^{34}\overline{z}^{28} + 3z^{34}\overline{z}^{26} - z^{33}\overline{z}^{35} + 7z^{33}\overline{z}^{33} + 12z^{33}\overline{z}^{31} + \ldots + 12z^{3}\overline{z}^{5} + 7z^{3}\overline{z}^{3} - z^{3}\overline{z} + 3z^{2}\overline{z}^{10} + 4z^{2}\overline{z}^{8} + 9z^{2}\overline{z}^{6} + 6z^{2}\overline{z}^{4} + 2z^{2}\overline{z}^{2} - z\overline{z}^{3} + 1\big) \Big/$$

$$\big((1-\overline{z}^6)(1-\overline{z}^4)(1-\overline{z}^4)(1-\overline{z}^2)(1-z^6)(1-z^4)(1-z^4)(1-z^2)(1-z^3\overline{z}^3)(1-z^2\overline{z}^2)^4(1-z\overline{z})(1-z^5\overline{z})(1-z^3\overline{z})^3(1-z^4\overline{z}^2)(1-\overline{z}^5z)(1-\overline{z}^3z)^3(1-\overline{z}^4z^2)\big)$$

$$= 1+z^2+z\overline{z}+\overline{z}^2+3z^4+3z^3\overline{z}+8z^2\overline{z}^2+3z\overline{z}^3+3\overline{z}^4+4z^6+6z^5\overline{z}+19z^4\overline{z}^2 +20z^3\overline{z}^3+19z^2\overline{z}^4+6z\overline{z}^5+4\overline{z}^6+7z^8+11z^7\overline{z}+47z^6\overline{z}^2+62z^5\overline{z}^3+98z^4\overline{z}^4 +62z^3\overline{z}^5+47z^2\overline{z}^6+11z\overline{z}^7+7\overline{z}^8+9z^{10}+18z^9\overline{z}+81z^8\overline{z}^2+150z^7\overline{z}^3 +278z^6\overline{z}^4+293z^5\overline{z}^5+278z^4\overline{z}^6+150z^3\overline{z}^7+81z^2\overline{z}^8+18z\overline{z}^9+9\overline{z}^{10} +14z^{12}+27z^{11}\overline{z}+143z^{10}\overline{z}^2+299z^9\overline{z}^3+669z^8\overline{z}^4+900z^7\overline{z}^5+1128z^6\overline{z}^6 +900z^5\overline{z}^7+669z^4\overline{z}^8+299z^3\overline{z}^9+143z^2\overline{z}^{10}+27z\overline{z}^{11}+14\overline{z}^{12} + \ldots$$

Four Qubits: Hilbert Series of $U(2)^{\otimes 4}$

$$H_U(z) = \big(z^{76}+6z^{70}+46z^{68}+110z^{66}+344z^{64}+844z^{62}+2154z^{60}+4606z^{58}+9397z^{56}$$
$$+16848z^{54}+28747z^{52}+44580z^{50}+65366z^{48}+88036z^{46}+111909z^{44}$$
$$+131368z^{42}+145676z^{40}+149860z^{38}+145676z^{36}+131368z^{34}$$
$$+111909z^{32}+88036z^{30}+65366z^{28}+44580z^{26}+28747z^{24}+16848z^{221}$$
$$+9397z^{20}+4606z^{18}+2154z^{16}+844z^{14}+344z^{12}+110z^{10}+46z^{8}+6z^{6}$$
$$+1\big) \Big/ \Big(\left(1-z^{10}\right)\left(1-z^{8}\right)^{4}\left(1-z^{6}\right)^{6}\left(1-z^{4}\right)^{7}\left(1-z^{2}\right)\Big)$$

$$= 1 + z^2 + 8z^4 + 20z^6 + 98z^8 + 293z^{10} + 1\,128z^{12} + 3\,409z^{14}$$
$$+ 10\,846z^{16} + 30\,480z^{18} + 84\,652z^{20} + 217\,677z^{22} + 544\,312z^{24}$$
$$+ 1\,289\,225z^{26} + 2\,961\,626z^{28} + 6\,528\,284z^{30} + 13\,980\,717z^{32}$$
$$+ 28\,963\,980z^{34} + 58\,464\,510z^{36} + 114\,806\,429z^{38} + \ldots$$

Four Qubits: Invariants of $U(2)^{\otimes 4}$

$$
\begin{aligned}
H_U(z) = {} & 1 + z^2 + 8z^4 + 20z^6 + 98z^8 + 293z^{10} + 1\,128z^{12} + 3\,409z^{14} \\
& + 10\,846z^{16} + 30\,480z^{18} + 84\,652z^{20} + 217\,677z^{22} + 544\,312z^{24} \\
& + 1\,289\,225z^{26} + 2\,961\,626z^{28} + 6\,528\,284z^{30} + 13\,980\,717z^{32} \\
& + 28\,963\,980z^{34} + 58\,464\,510z^{36} + 114\,806\,429z^{38} + \ldots
\end{aligned}
$$

intermediate results:

1 invariant of degree 2	these 109 invariants generate a (sub)ring with series $1 + z^2 + 8z^4 + 20z^6 + 98z^8 + 221z^{10} + \ldots$
7 invariants of degree 4	
12 invariants of degree 6	
50 invariants of degree 8	
39 invariants of degree 10	

$\Longrightarrow$ even more invariants are required to generate the whole invariant ring

Relation Ideal

Problem:
Given some invariants $f_1, \ldots, f_m$, do they generate the full invariant ring?

evaluation homorphism: $\mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}[x_1, \ldots, x_d]$

$$g(y_1, \ldots, y_m) \mapsto g(f_1, \ldots, f_m)$$

relation ideal:

$$\operatorname{Rel}(f_1, \ldots, f_m) = \{g(y_1, \ldots, y_m) \colon g(f_1, \ldots, f_m) = 0\} \trianglelefteq \mathbb{K}[y_1, \ldots, y_m]$$

$$\mathcal{A} = \langle f_1, \ldots, f_m \rangle \cong \mathbb{K}[y_1, \ldots, y_m] / \operatorname{Rel}(f_1, \ldots, f_m)$$

Hilbert series: $\operatorname{Hilb}(\mathcal{A}) = \operatorname{Hilb}(\operatorname{Rel})$

computed (in principle) as

$$\operatorname{Rel}(f_1, \ldots, f_m) = \langle f_1 - y_1, \ldots, f_m - y_m \rangle \cap \mathbb{K}[y_1, \ldots, y_m]$$

SAGBI Bases

Subalgebra **A**nalogue to **G**röbner **B**asis for **I**deals[a]

- basis $B = \{g_1, \dots, g_\ell\}$ of a subalgebra $\mathcal{A} = \langle f_1, \dots, f_m \rangle \subset \mathbb{K}[x_1, \dots, x_n]$
- depends on a term ordering $>$ for polynomials, *e. g.*, lexicographic ordering $x_1 > x_2 > \cdots > x_n$
- the semigroup $\mathrm{LM}(\mathcal{A})$ of leading monomials of $\mathcal{A}$ is generated by $\mathrm{LM}(B)$, *i. e.* $\mathrm{LM}(\mathcal{A}) = \langle \mathrm{LM}(g_1), \dots, \mathrm{LM}(g_\ell) \rangle$
- allows membership test for $\mathcal{A}$ via top reduction:

$$h \xrightarrow{B} h - c g_{i_1}^{e_1} \cdots g_{i_k}^{e_k} \quad \text{if } \mathrm{LT}(h) = c\,\mathrm{LT}(g_{i_1})^{e_1} \cdots \mathrm{LT}(g_{i_k})^{e_k}$$

- need not be finite, even if $\mathcal{A}$ is finitely generated

[a] Kapur & Madlener 1989, Robbiano & Sweedler, 1990

Computing SAGBI Bases

semi-algorithm to compute a SAGBI basis

0. set $B \leftarrow \{f_1, \ldots, f_m\}$
1. compute the relation ideal $\mathrm{Rel}(\mathrm{LM}(B))$ of the leading monomials of B
2. for all generators $r(y_1, \ldots, y_m)$ of $\mathrm{Rel}(\mathrm{LM}(B))$, compute $r(f_1, \ldots, f_m) \xrightarrow{B} h$
3. if $h \neq 0$, add h to B
4. repeat from Step 1. until no new element has been added to B

Computing SAGBI Bases

semi-algorithm to compute a SAGBI basis

0. set $B \leftarrow \{f_1, \ldots, f_m\}$
1. compute the relation ideal $\mathrm{Rel}(\mathrm{LM}(B))$ of the leading monomials of B *up to degree* d
2. for all generators $r(y_1, \ldots, y_m)$ of $\mathrm{Rel}(\mathrm{LM}(B))$, compute $r(f_1, \ldots, f_m) \xrightarrow{B} h$
3. if $h \neq 0$, add h to B
4. repeat from Step 1. *with increased bound* d until no new element has been added to B

Using SAGBI Bases

assume $B = \{g_1, \ldots, g_\ell\}$ is a SAGBI basis of the polynomial algebra $\mathcal{A}$

all relevant information is given by the leading monomials

- $\mathrm{Hilb}(\mathcal{A}) = \mathrm{Hilb}(\langle \mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_\ell) \rangle)$
- the Hilbert series can be computed from the ideal

$$\mathrm{Rel}(\mathrm{LM}(B)) = \langle \mathrm{LM}(g_1) - t_1, \ldots, \mathrm{LM}(g_\ell) - t_\ell \rangle \cap \mathbb{K}[t_1, \ldots, t_\ell]$$

- if B has been been computed only up to degree d, we can still compare the Hilbert series

$\Longrightarrow$ direct proof of completeness for two-qubits mixed state < 1 min

$\Longrightarrow$ proof of completeness for $SU(2)^{\otimes 3}$
("private communication" in Luque, Thibon & Toumazet (2007))

Three Qubits

(joint work with Robert Zeier, work in progress)

- action of $U(2)^{\otimes 3}$ on density matrices ρ (or Hamiltonians) via conjugation
- adjoint representation of $SU(2)$ decomposes as $1 \oplus 3$
  $\Longrightarrow (1 \oplus 3)^3 = 1 \oplus 3 \times 3 \oplus 3 \times 3^2 \oplus 3^3$
- corresponds to the action on

$$\begin{gathered} I_2 \otimes I_2 \otimes I_2 \\ \oplus\, (\mathfrak{su}(2) \otimes I_2 \otimes I_2) \oplus (I_2 \otimes \mathfrak{su}(2) \otimes I_2) \oplus (I_2 \otimes I_2 \otimes \mathfrak{su}(2)) \\ \oplus\, (\mathfrak{su}(2) \otimes \mathfrak{su}(2) \otimes I_2) \oplus (\mathfrak{su}(2) \otimes I_2 \otimes \mathfrak{su}(2)) \oplus (I_2 \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)) \\ \oplus\, (\mathfrak{su}(2) \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)) \end{gathered}$$

- invariant ring (excluding the trivial rep.) admits 7-fold grading
- Hilbert series $H(z_1, z_2, z_3,\ z_{12}, z_{13}, z_{23},\ z_{123})$
- consider only some of the irreducible components

Three Qubits: Partial Results

univariate Hilbert series

$$
\begin{aligned}
H(z) &= (z^{206} + \ldots + 1)/(1 - \ldots - z^{270}) \\
&= 1 + z + 8\,z^2 + 24\,z^3 + 148\,z^4 + 649\,z^5 + 3.576\,z^6 + 17.206\,z^7 \\
&\quad + 84.320\,z^8 + 386.599\,z^9 + 1.720.880\,z^{10} + 7.302.550\,z^{11} + 29.864.124\,z^{12} \\
&\quad + 117.329.840\,z^{13} + 444.769.448\,z^{14} + 1.627.560.935\,z^{15} + \ldots
\end{aligned}
$$

computed up to degree 8000 in about 4.5 days via two-fold integration and (Laurent) series expansion using `LazySeries` in MAGMA

Three Qubits: Partial Results

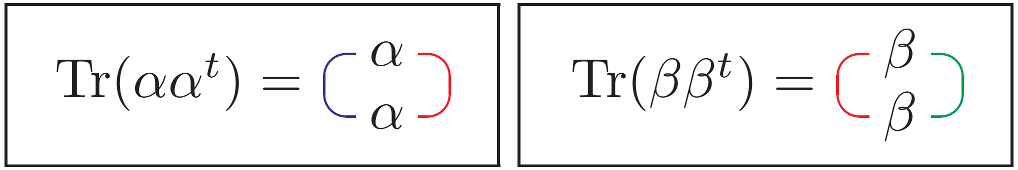
action on two components of dimension 9

- Hilbert series

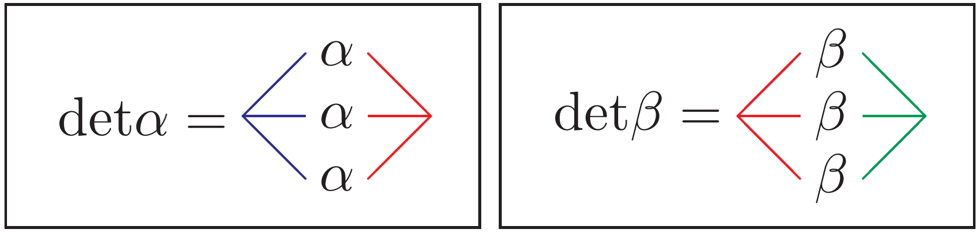
$$H_{9\oplus 9}(z) = \frac{1 + z^8 + z^{16}}{(1-z^2)^2(1-z^3)^2(1-z^4)^3(1-z^6)^2}$$
$$= 1 + 2\,z^2 + 2\,z^3 + 6\,z^4 + 4\,z^5 + 15\,z^6 + 12\,z^7 + 31\,z^8 + 28\,z^9$$
$$+ 62\,z^{10} + 58\,z^{11} + 120\,z^{12} + 112\,z^{13} + 213\,z^{14} + 212\,z^{15}$$
$$+ 370\,z^{16} + 368\,z^{17} + 622\,z^{18} + 628\,z^{19} + 1006\,z^{20} + \ldots$$

- generated by 9 primary invariants and 1 additional invariant
- completeness follows from the fact that there is only one additional invariant

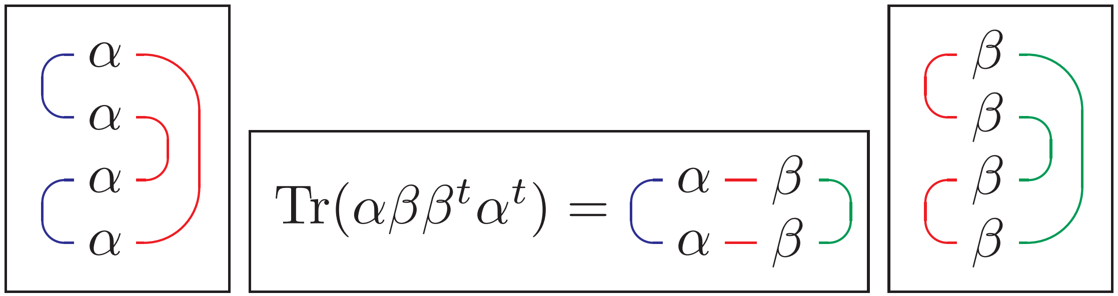
degree 2:



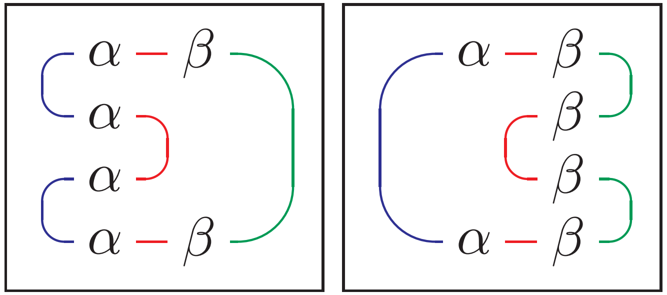
degree 3:



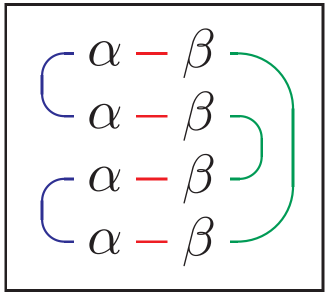
degree 4:



degree 6:



degree 8:



Are there any relations for these tensors like Cayley-Hamilton?

Three Qubits: Partial Results

action on three components of dimension 9

$$
\begin{aligned}
H_{3\times 9}(z) = \big(&z^{36} - z^{35} - z^{34} + z^{33} + 4\,z^{32} + 6\,z^{30} - 2\,z^{29} + 12\,z^{28} + 12\,z^{27} + 33\,z^{26}\\
&+ 28\,z^{25} + 69\,z^{24} + 45\,z^{23} + 82\,z^{22} + 73\,z^{21} + 116\,z^{20} + 86\,z^{19} + 134\,z^{18}\\
&+ 86\,z^{17} + 116\,z^{16} + 73\,z^{15} + 82\,z^{14} + 45\,z^{13} + 69\,z^{12} + 28\,z^{11} + 33\,z^{10}\\
&+ 12\,z^{9} + 12\,z^{8} - 2\,z^{7} + 6\,z^{6} + 4\,z^{4} + z^{3} - z^{2} - z + 1\big)/\\
&\big((z-1)^{18}(z+1)^{11}(z^2-z+1)^2(z^2+1)^5(z^2+z+1)^6(z^4+z^3+z^2+z+1)^2\big)\\
= {}& 1 + 3\,z^2 + 4\,z^3 + 15\,z^4 + 18\,z^5 + 63\,z^6 + 90\,z^7 + 240\,z^8 + 386\,z^9 + 882\,z^{10}\\
&+ 1.479\,z^{11} + 3.093\,z^{12} + 5.247\,z^{13} + 10.179\,z^{14} + 17.299\,z^{15} + 31.695\,z^{16}\\
&+ 53.133\,z^{17} + 93.143\,z^{18} + 153.354\,z^{19} + 258.852\,z^{20} + \ldots
\end{aligned}
$$

- computed 178 invariants with max. degree 12
- verified up to degree 20 using triple-grading, max. dimension 6.281