Mathematical Circus

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Contents of Akiyama's Talk

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In the lectures I give all over Japan, I have always tried to give the audience an unforgettable experience of mathematics. This is accomplished through a careful choice of relevant topics such as the applications of mathematics in daily life and also through the use of specially crafted hands-on models that help to demonstrate the beauty of mathematics. In this talk, I will discuss some of these choice topics and present some of those models, namely:

1. Mathematics in Music

a) La Galerian musical performance

b) Three harmonies:
do-mi-sol, do-fa-la, si-re-sol

c) The Pythagorean Theorem

d) Listen to the sound of Numbers
Finite decimals
Infinite repeating decimal
Infinite decimal

e) A spiral xylophone

f) CD

Models and Experiments

a) Two music scales
Accordion

b) Circular scales
A loop marked with 12 equal distances

c) Slide Type
Three elephants

d) Organite (big size)
Spanish Song (punch cards)
1/8, 5/7, π

e) A spiral xylophone

f) CD player and CD

2. An Important Property of the Reuleaux Triangle

a) Manhole covers
b) Square drill and hexagonal drill
c) Vehicle wheels with constant width
d) Rotary engines

a) Manhole covers
b) Square & hexagonal drills
c) Vehicle wheels with constant width
d) Rotary engines

3. Applications of Conic Sections

a) Ellipse: ESWL
b) Parabola: Solar cookers
c) Hyperbola: Gears

a) Ellipse bowl
b) Parabola and ping pong balls
Solar cookers
c) Twisted cylinder
Gears

4. Think Your Way through Math

a) Area of a circle
b) Surface of a sphere
c) Volume of a sphere
d) Volume of a rhombic dodecahedron

a) Baumkuhen type
b) Onion slice model finding $4\pi r^2$
c) Water Melon
d) Fox-snake (rhombic dodecahedron↔ box)
Reversible Solid
(truncated octahedron ↔box)
Tool peeling skin apples
Apples

1. Mathematics in music

a) La Galerian musical performance

b) Three harmonies

In a circular scale (Figure 1.1) the twelve semitones in music, do, do#, re, re#, mi, mi#, fa, sol, sol#, la, la#, si are arranged in a balanced circle. In the three harmonies, do-mi-sol, do-fa-la, si-re-sol, the distances between the three notes are 4-3-5, 5-4-3 and 3-5-4, respectively.

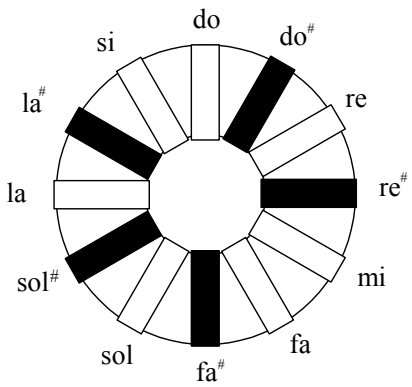


Figure 1.1

c) The Pythagorean Theorem

Then you will find that all distances have the sequence 3-4-5.

What do you remember of the ratio 3:4:5 ? Form a piece of string into a loop and stretch it out by holding the three points which divide the string in the ratio 3:4:5. The result is shown in Figure 1.2. The ratio 3:4:5 is a typical ratio for sides of right triangles.

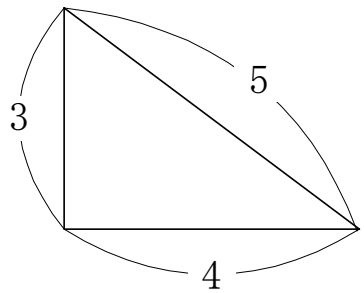


Figure 1.2

Besides 3:4:5, the ratio 5:12:13 is also the ratio for sides of a right triangle. There are infinitely many such ratios.

Let x, y, z be sides of a right triangle where z is the hypotenuse. Then the Pythagorean theorem states $x^2 + y^2 = z^2$. The model (Figure 1.3) provides a visual illustration of the Pythagorean Theorem. It has three thin containers with square cross sections mounted on a circular base. The dimensions of the squares are determined by the Pythagorean Theorem. Initially, each of the smaller containers is filled with pieces of plastic. As the circular base rotates, the pieces from the smaller containers fall into the larger container and fill it up exactly.

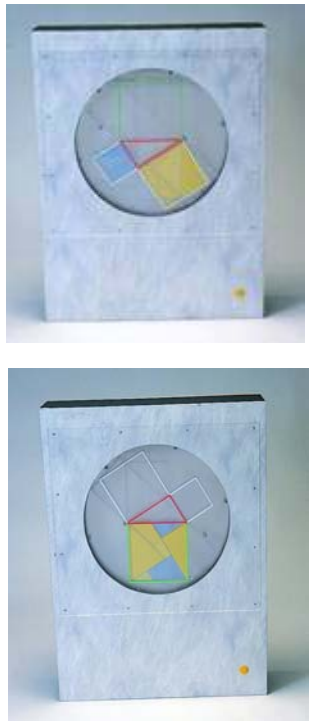


Figure 1. 3

Instead of squares let's take any three similar figures, for example, the elephants in Figure 1.4. They can be used to illustrate as long as the ratios of lengths of their sides equal those of a right triangle. By using a balance, we can show that the weight of the two small elephants equals the weight of the large elephant when the thickness is the same.

Figure 1.4

d) The Sound of Numbers

An Organite(Figure 1.5) produces music using a paper sheet with punched holes. We can hear the music of numbers. Let 0(zero) correspond to "do", 1 to do#, "2" to re, …,and so on. What is the melody for .125(1/8)? The music has only three notes and it is very short! How about .714285714285714285…(5/7) ? It is an infinite repeating decimal, so its melody is repeated. Such numbers are called rational numbers. Now π (the circular constant) is 3.1415…., and there is no repeated pattern in its infinite decimal part. So the melody is endless without repeating. Such numbers are called irrational numbers.

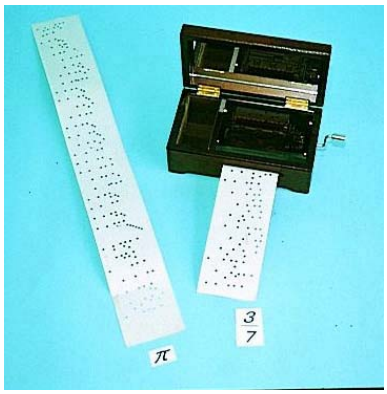


Figure 1.5

e) A Spiral Xylophone

Figure 1.6 is a spiral xylophone. A wooden ball from the top of the spiral rolls down and plays music! The length of a key determines its tone, and the spacing of the keys determines the rhythm. Pauses are caused by keys that are covered with felt. Please enjoy the music!

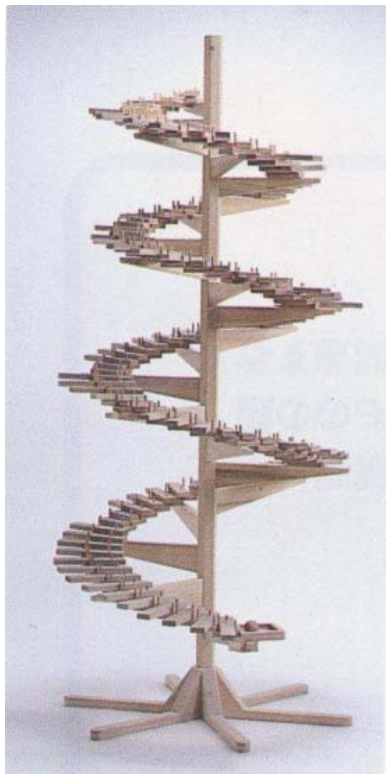


Figure 1.6

f) CD

Let's listen to music from a CD. Those are beautiful songs. Let's make a few scratches from the center of the CD to the edge with a sharp pin (Figure 1.7), then play the CD again. Surprise! There is no difference! We can still hear the music clearly.

How are CD's protected from scratches? Some kind of mathematics is at work – the principle behind error correcting codes.

Figure 1.7

2. An Important Property of the Reuleaux Triangle

a) Manhole Covers

Why are manhole covers round?

A square cover can fall into a square manhole of the same size (Figure 2.1). This is because the diagonal of a square is longer than the length of its sides. On the other hand, a round cover cannot fall into a round manhole of the same size because a circle has constant width, i.e., the distance between any two parallel lines which are tangent to the circle is constant.

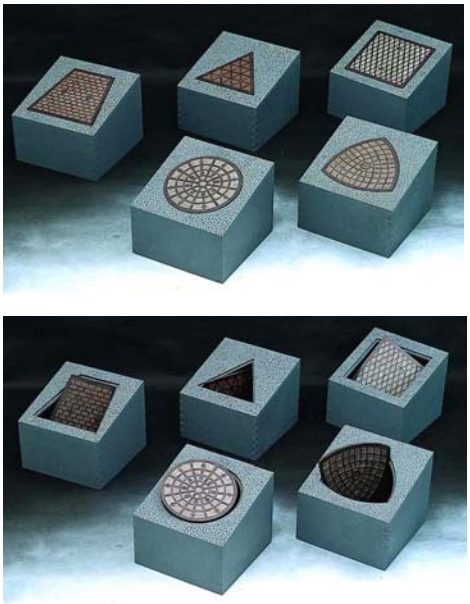


Figure 2.1

Other figures of constant width can also be used successfully as manhole covers. One such figure is the *Reuleaux triangle* (the lower right manhole cover in Figure 2.1). To draw a Reuleaux triangle, start with an equilateral triangle. Then, using one vertex of the triangle as a center and a side of the triangle as a radius, draw a circular arc of 60 degrees. Do the same using each vertex in turn (Figure 2.2).

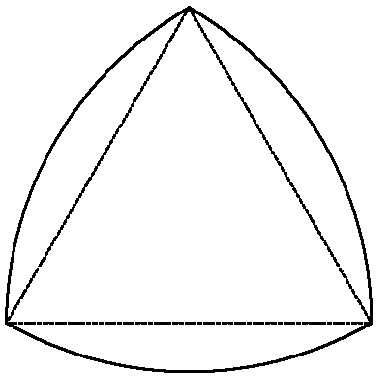


Figure 2.2

There are infinitely many figures of constant width, among them are Reuleaux pentagons, Reuleaux heptagons, etc.

b) Square Drill and Hexagonal Drill

A standard drill makes a round hole, but the drill in Figure 2.3a makes a square hole. The

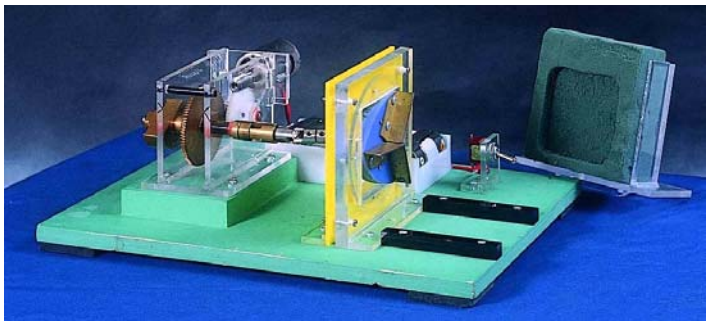


Figure 2.3a

blades are patterned after a Reuleaux triangle. A flexible axis allows the blades to rotate within a space confined by the circumference of a square as shown in Figure 2.3b. A square hole results.

Figure 2.3b

This drill (Figure 2.4a) has blades patterned after a Reuleaux pentagon. The axis is flexible as

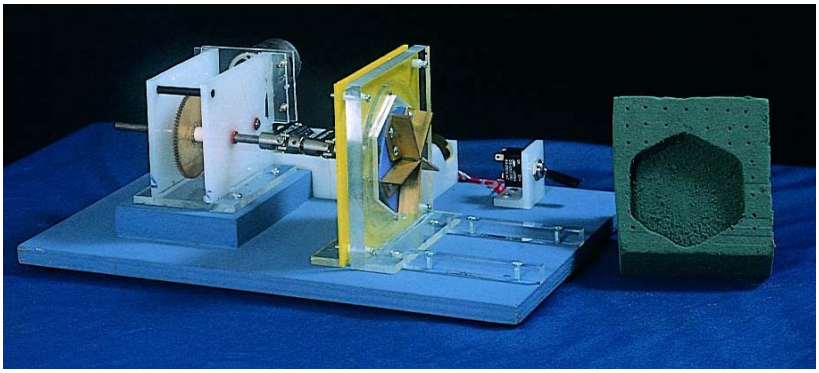


Figure 2.4a

with the square drill. The blades rotate within a space confined by the circumference of a hexagon to form a hexagonal hole(Figure 2.4b).

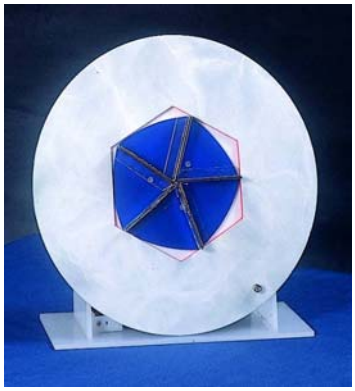


Figure 2.4b

c) Vehicles with Wheels of Constant Widths

A board placed on top of two logs is sometimes used to move heavy loads. The circular cross-sections of the logs allow them to roll smoothly. The same can be said about rollers whose cross-sections are other figures with

Figure 2.5a

Figure 2.5b

constant width as is demonstrated by these devices(Figure 2.5a & b).

The wagon shown in Figure 2.6 has wheels, which are warped, but the carriage moves smoothly in the horizontal direction because the wheels have constant width. If ordinary axles were attached to these wheels, then they go up and down as the wheels rotate and the carriage will not move horizontally. In this wagon, axles are made of boards with the shape of Reuleaux triangles and these Reuleaux triangles within square frames circumscribe them. Therefore, the carriage,which is attached to these square frames, always remains the same distance away from the ground when the axles rotate. Consequently, the carriage moves horizontally. This wagon was invented by the Mexican high school students, Sebastian von Wuthenau Mayer and Claudia Masferrer Leon.

Figure 2.6

d) Rotary Engine

In most engines, a piston which moves up and down triggers the four mechanical jobs of gas intake→compression→combustion→exhaustion. In a rotary engine, however, it is a rotating triangular rotor that causes these four functions.

There are two important components for rotary engines: a broad bean-like housing, and a triangular shaped rotor. The rotor rotates inside

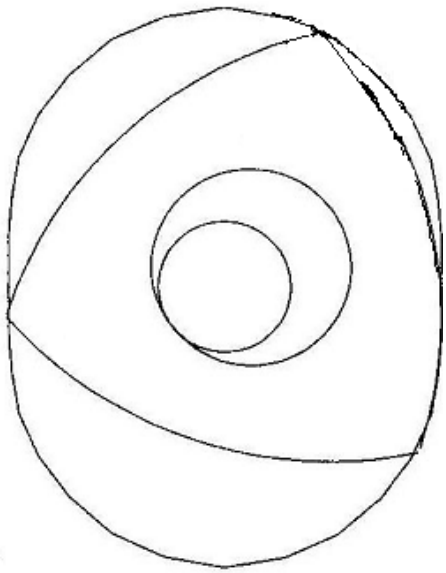
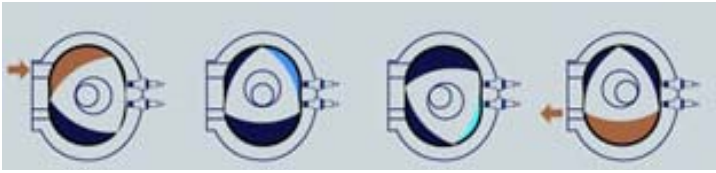


Figure 2.7

the housing, its three vertices touching the housing all the time (Figure 2.7). The space between the housing and the rotor is divided into three chambers and the volume of each chamber changes as the rotor turns.

At first, the intake of gas increases the volume of one chamber. As the rotor moves, this volume decreases. The change in the volume of gas makes intake→compression→explosion→exhaustion possible (Figure 2.8).



Intake Compression Explosion Exhaustion

Figure 2.8

3. Applications of Conic Sections

a) Ellipse : Extracorporeal shock wave lithotripsy (ESWL)

ESWL is a modern medical procedure often used to shatter kidney stones without surgery. In this treatment, shock waves are generated outside the body of the patient (hence, the term "*extracorporeal*") and are delivered to the kidney stones via an ellipsoidal reflecting medium. The intensity of the shock waves causes fragmentation of the stones.

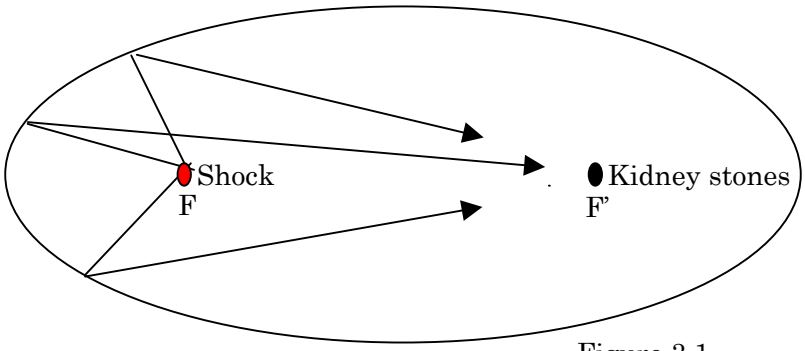


Figure 3.1

The principle behind ESWL is that of the properties of the ellipse. An ellipse is a set of points on the plane, the sum of whose distances from two fixed points F and F', called foci, is a constant. In ESWL, shock waves are generated from one focus and travel through the ellipsoidal medium. The waves are then reflected to the other focus where the kidney stone is located(Figure 3.1).

b) Parabola

A *parabola* consists of a set of points which are equidistant from a point called the *focus* and a line called the *directrix* (Figure 3.2).

It is easily verified that a ray parallel to the axis of the parabola will always be reflected to the focus once it hits the parabola (Figure 3.2). The device shown in Figure 3.3 makes use of this property. It shoots balls from a rail which is perpendicular to the

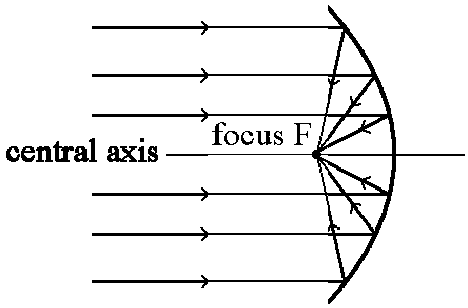


Figure 3.2

axis of a parabola that serves as a reflecting board. The balls are shot in the direction of the parabola and are always reflected to the hole which is located at the focus of the parabola.

Figure 3.3

A parabolic antennae concentrates light at its focus. It is interesting to watch such an antenna use the sun's rays to bake a potato located at its focus(Figure 3.4).

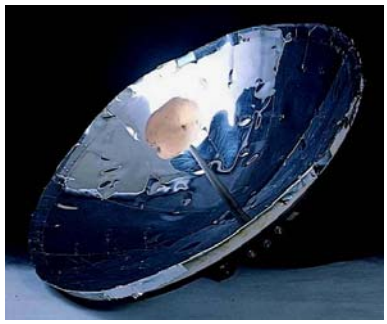


Figure 3.4

c) Hyperbola: Gears

The first model (Figure 3.5) is a cylindrical framework which is converted, with one twist, into a hyperboloidal framework. The second model (Figure 3.6) is a differential gear which transmits motion. The two parts of the gear are hyperboloidal [HC].

Figure 3.5

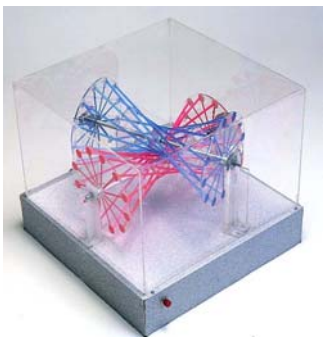


Figure 3.6

4. Think your Way through Math

a) Area of a circle

How can you explain the area of a circle with the radius r is πr^2? Figures 4.1 and 4.2 show how the area of a circle can be approximated by the area of an isosceles triangle whose base is equal to the length of the whole circular arc and whose height is the radius. Hence the area of the circle is $\frac{1}{2}\times(base)\times(height)$ of the isosceles triangle, or $\frac{1}{2}\times 2\pi r\times r=\pi r^2$.

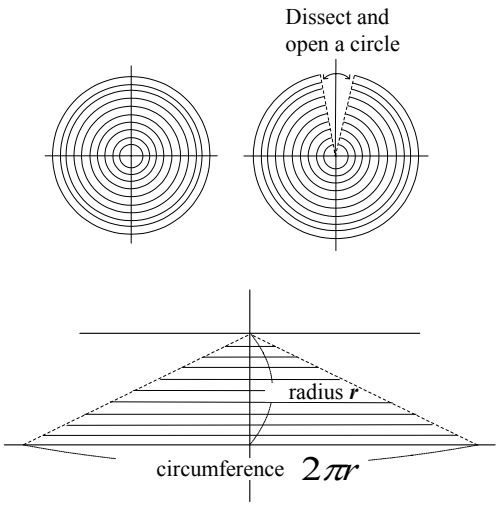


Figure 4.1

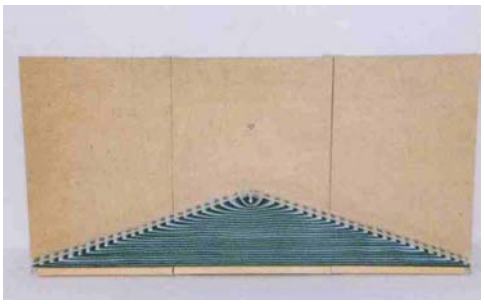


Figure 4.2

b) Surface Area of a Sphere

The surface of a sphere is approximated by the following device. A length of plastic tube is used to line the surface of a hemisphere. The tube extends to line the interior of two circles with a radius equal to the radius of the hemisphere, say r (Figure 4.3). Colored liquid is pumped into the tube until the length covering the entire surface of the hemisphere is filled. The liquid is then made to descend to the two circles. It is at the point that the hemisphere empties completely when the two circles are completely filled. This observation leads to the following conclusion:

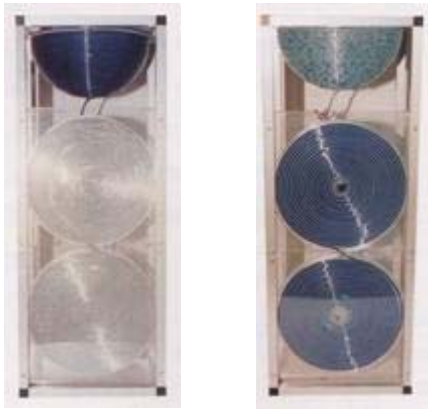


Figure 4.3

Surface area of a hemisphere

$= 2 \times$ (area of the circle)

or

Surface area of a sphere

$= 4 \times$ (area of the circle) $= 4\pi r^2$.

c) Volume of a Sphere

The volume of a sphere can be approximated by packing the sphere with cones whose heights are equal to the radius of the sphere, say r (Figure 4.4). A device based on this idea is shown in Figure 4.5. Thus

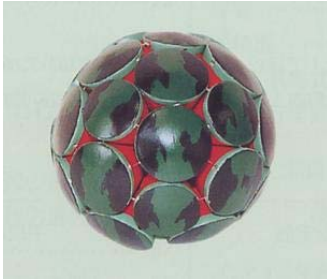


Figure 4.4

Volume of sphere

$$= \sum(\text{volumes of all cones})$$

$$= \sum \frac{1}{3} \times (\text{height of cone}) \times (\text{base area of cone})$$

$$= \frac{1}{3} \times r \times \sum(\text{base areas of all cones})$$

$$= \frac{1}{3} \times r \times (\text{surface area of sphere})$$

$$= \frac{1}{3} \times r \times 4\pi r^2 = \frac{4}{3}\pi r^3$$

Figure 4.5

d) Volume of a rhombic dodecahedron

How to Construct a Fox-Snake

To begin with, in order to explain the structure of a rhombic dodecahedron, we construct a solid with double crosses by using 7 cubes of the same size as shown in Figure 4.6. If we take one of these cubes and dissect it, as shown in Figure 4.7, by means of 6 planes going through the center of the cube, then 6 pyramids (pentahedra with a square base) will be obtained. If we perform this dissection to each of the 6 cubes other than the central cube of the solid with double crosses of Figure 4.6, and discard those pyramids which are not touching face-to-face with the central cube of the solid with double crosses, then we end up with the rhombic dodecahedron of Figure 4.8.

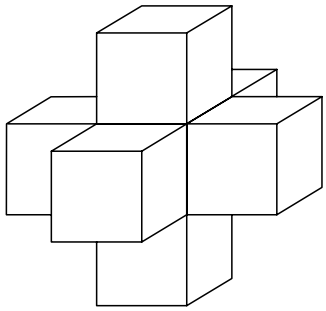


Figure 4.6

From the construction above it is clear that a rhombic dodecahedron has 14 vertices, consisting of 8 vertices of the central cube and 6 “centers” of the surrounding pyramids (the “center” of a pyramid being the point of intersection of the 4 edges of the pyramid). We can also see that if we partition the space into adjacent cubes and color these cubes black and white in a 3-dimensional chess-board pattern, and then place these rhombic dodecahedrons in space by putting the central cube of each of them on the spot where a black cube is located, then the entire space can be filled, i.e., rhombic dodecahedrons are space-filling solids. Furthermore, we see that the volume of a rhombic dodecahedron is twice that of the central cube, and therefore, is the same as the volume of the rectangular parallelepiped shown in Figure 4.9.

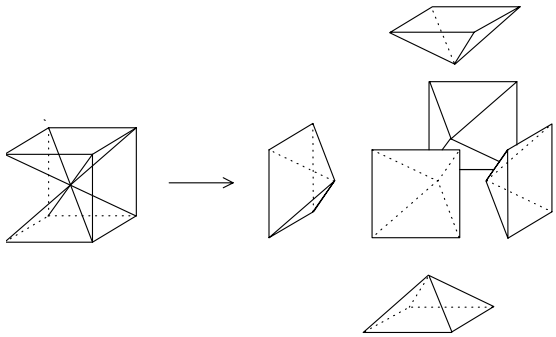


Figure 4.7

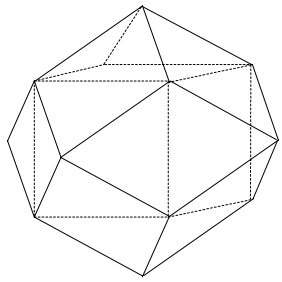


Figure 4.8

Next, if we dissect the rhombic dodecahedron of Figure 4.8 by means of 3 planes parallel to the faces of the central cube and intersecting at the center of the cube, then the rhombic dodecahedron is partitioned into 8 congruent hexahedrons. Figure 4.10 (a) shows one of these hexahedrons and Figure 4.10 (b) shows its development. If we put together 4 of these hexahedrons in such a way that the vertices, corresponding to the vertex A of Figure 4.10, of the four hexahedrons come together, then a cube will result. Figure 4.11 illustrates this fact: l

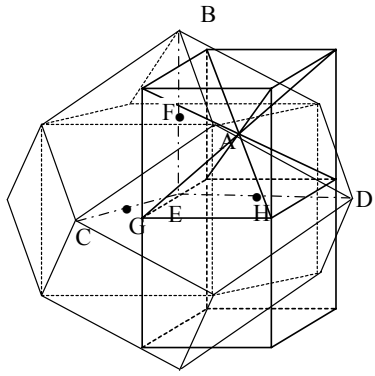


Figure 4.9

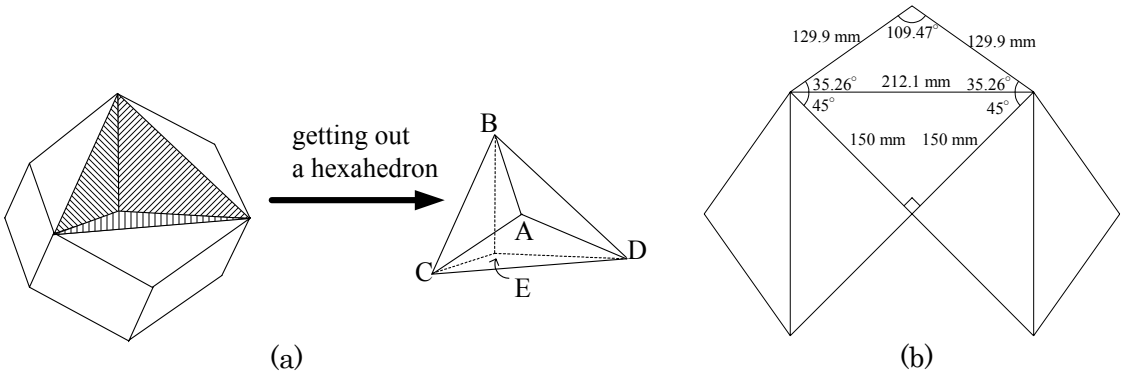


Figure 4.10

lengths of the 3 line segments EB, EC and ED are all equal to the length of an edge of the resulting cube, and these line segments are pair-wise perpendicular. Also, the points F, G, H in Figure 4.11 are the mid-points of the segments EB, EC and ED, respectively, and they lie on 3 different faces of the resulting cube. Since we know that the rhombic dodecahedrons of Figure 4.8 are space filling, we can conclude that 4 hexahedrons will yield a cube if they are glued in such a way that the vertex of each one, corresponding to the vertex A, come together.

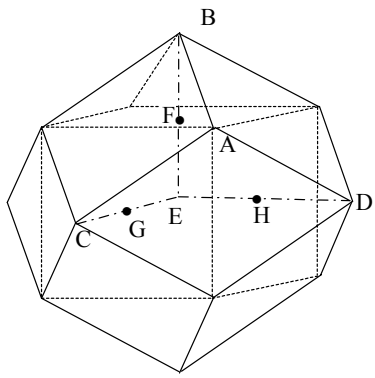
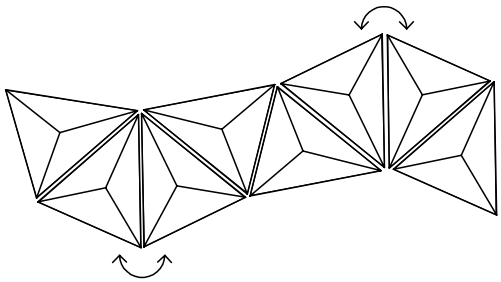


Figure 4.11

Now that we know that a cube will result by putting together 4 hexahedrons of Figure 4.10 in a specific way, we see that the rectangular parallelepiped of Figure 4.9 can still be constructed from an aggregate of 8 hexahedrons resulting from a dissection of a rhombic dodecahedron, hinged together as shown in Figure 4.12. This shows that a rhombic dodecahedron of Figure 4.8 and rectangular parallelepiped of Figure 4.9 can be obtained from one another by "jointed turning inside out." Furthermore, two pairs of hexahedrons indicated by arrows in Figure 4.12 are joined together invariantly throughout the process of "turning inside out", and therefore, it is unnecessary to dissect each of these pairs into 2 hexahedrons, respectively. This fact enables us to obtain an even more elegant method of obtaining dissections turning a rhombic dodecahedron into a rectangular parallelepiped and vice versa. Yasuyuki Yamaguchi, a plastic artist, created an artistic object called "fox-snake", based on the idea of this "jointed turning inside out", by painting a fox on the surface of a rhombic dodecahedron and a rat snake on that of the rectangular parallelepiped obtained by turning inside out (see Figure 4.13). With this object you notice, by pulling a string, a fox gets gobbled up instantly by a rat snake. Having seen this amusing object, we decided in our seminar to call



Figure

this model of “jointed turning inside out” the “fox-snake” model.

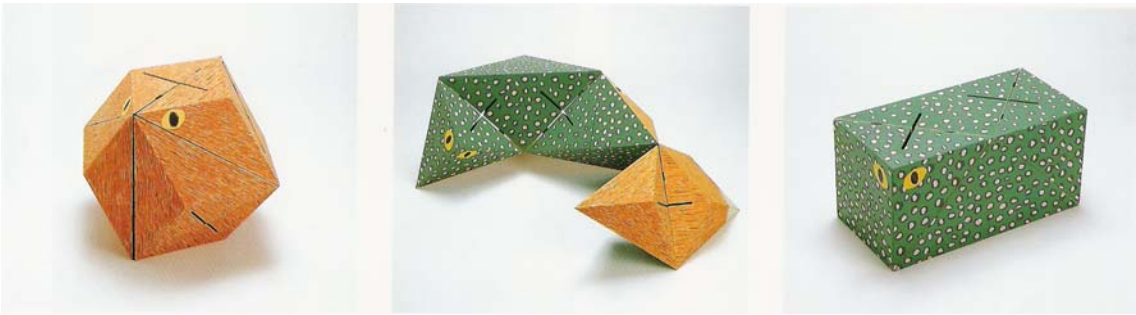


Figure 4.13

The reason why a rhombic dodecahedron (Figure 4.8) and a rectangular parallelepiped (Figure 4.9) can be obtained from each other by the process of “jointed turning inside out” can be explained more theoretically by using the fact both of these solids are space filling solids; however, we omit the explanation since a graphic description of 3 dimensional objects gets very complicated.

References

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