Solutions of the Laplacian flow and coflow of a Locally Conformal Parallel $G_2$-structure

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Outline

1. Motivation and $G_2$ background.
2. Laplacian flow and coflow.
3. Laplacian flow and coflow of an LCP $G_2$-structure.
4. Results.
The holonomy group of a linear connection

Suppose $M^m$ simply connected.

Let $\nabla$ be a linear connection on $M$, $p \in M$ and $\gamma: [0, 1] \rightarrow M$ a curve such that $\gamma(0) = \gamma(1) = p$.

The parallel transport along $\gamma$ defines an endomorphism:

$$P_\gamma: T_p M \rightarrow T_p M.$$ 

The holonomy group $Hol_p(\nabla)$ of $\nabla$ based at $p$:

$$\{P_\gamma\} \subset GL(m, \mathbb{R}).$$

For all $p, q \in M$, $Hol_p(\nabla)$ is conjugated to $Hol_q(\nabla)$:

the holonomy group $Hol(\nabla)$ of $\nabla$.

In particular, if $g$ is a Riemannian metric and $\nabla = \nabla^g$ is the Levi-Civita connection, then $Hol(\nabla^g) \subset O(m)$ ($SO(m)$ if $M$ is orientable).
Riemannian holonomy

[ Berger'55]: Some of the possible holonomy groups of a Riemannian, simply connected, irreducible and nonsymmetric \((M, g)\) are:

\[
Hol(\nabla^g) \subseteq SU(n) \quad \text{in dimension } m = 2n \quad \text{(Calabi-Yau)};
\]

\[
Hol(\nabla^g) \subseteq G_2 \quad \text{in dimension } m = 7;
\]

\[
Hol(\nabla^g) \subseteq \text{Spin}(7) \quad \text{in dimension } m = 8.
\]

In these cases, the metric \(g\) is \text{Ricci flat}.

Question: “Are there Riemannian metrics \(g\) with special \(Hol(\nabla^g)\)?”

A 7-dimensional manifold $M$ has a $G_2$-structure $\iff$ exists a global 3-form (fundamental form) $\sigma \in \Omega^3(M)$ having the local expression

$$\sigma = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

where $\{e^1, \ldots, e^7\}$ is a local coframe and $e^{ij}$ stands for $e^i \wedge e^j$.

- **The metric induced by $\sigma$:**

$$g(X, Y)vol = \frac{1}{6} \iota_X \sigma \wedge \iota_Y \sigma \wedge \sigma$$

for any $X, Y \in \mathfrak{X}(M)$.

- **Volume form:**

$$vol = e^{1234567}$$

- **The 4-form $\psi = *\sigma$:**

$$\psi = *\sigma = e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}$$
Fernández-Gray classification

[Fernández-Gray’82]

\[ \nabla^{LC} \sigma \in \Omega^1 \otimes \Omega^3_7 = X. \]

Under the action of the group \( G_2 \), \( X \) can be decomposed into 4 irreducible components

\[ X = X_1 \oplus X_2 \oplus X_3 \oplus X_4 \]

Therefore, there exist 16 different classes of \( G_2 \)-structures, called the Fernández-Gray classes. Some examples:

<table>
<thead>
<tr>
<th>Type</th>
<th>Class</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallel</td>
<td>( \mathcal{P} )</td>
<td>( d\sigma = d\ast\sigma = 0 )</td>
</tr>
<tr>
<td>Calibrated</td>
<td>( X_2 )</td>
<td>( d\sigma = 0 )</td>
</tr>
<tr>
<td>Cocalibrated</td>
<td>( X_1 \oplus X_3 )</td>
<td>( d(\ast\sigma) = 0 )</td>
</tr>
<tr>
<td>Locally Conformal Parallel</td>
<td>( X_4 )</td>
<td>( d\sigma = 3\tau_1 \wedge \sigma, ) ( d(\ast\sigma) = 4\tau_1 \wedge (\ast\sigma) )</td>
</tr>
</tbody>
</table>

\( \tau_1 \in \Omega^1(M) \).
Laplacian flow of a $G_2$-structure

[Bryant’06] introduced the Laplacian flow (LF)

$$\frac{d}{dt} \sigma_t = \Delta_t \sigma_t$$

where $\Delta$ is the Hodge Laplacian $\Delta = \delta d + d\delta$, with $\delta : \Omega^p \rightarrow \Omega^{p-1}$ such that $\delta = (-1)^p \ast d \ast$.

Properties:

- $\{ \sigma_t \text{ solution of (LF)} \}$
- $\sigma_0 \text{ calibrated } (d\sigma_0 = 0) \implies \sigma_t \text{ also calibrated.}$

- Laplacian flow can be considerate as the gradient flow of the Hitchin’s volume functional.

- If there exist solution, it converges to a torsion-free (parallel) $G_2$-structure, i.e. $\text{Hol}(\nabla^{g_{\infty}}) = G_2$.

Results:

- [Bryant-Xu’11] Short time existence and uniqueness of solution for compact manifolds (DeTurck’s Trick).

- [Lotay-Wei’15] Long time existence of solution starting near a torsion free structure.

- [Fernández-Fino-Manero’16] First examples of long time existence of solution.
Solutions of the Laplacian flow and coflow of an LCP $G_2$-structure

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Motivation and $G_2$ background
Laplacian flow and coflow
Laplacian flow and coflow of a LCP $G_2$-structure
Results

Laplacian coflow of a $G_2$-structure

[Karigiannis-McKay-Tsui’12] introduced the Laplacian coflow (LcF)

$$\frac{d}{dt} \psi_t = -\Delta_t \psi_t$$

where $\psi = \ast \sigma$.

Properties:

- $\sigma_t$ solution of (LcF)
  - $\sigma_0$ cocalibrated ($d \ast \sigma_0 = 0$) $\implies \sigma_t$ also cocalibrated.
- If there exist solution, it converges to a torsion-free (parallel) $G_2$-structure, i.e. $\text{Hol}(\nabla_{g_\infty}) = G_2$.

Results:

- Short time existence and uniqueness of solution is not known.
- [Grigorian’13] Introduced the modified Laplacian coflow and proved short time existence and uniqueness of solution.
- [Bagaglini-Fernández-Fino’17] First examples of long time existence of solution for coflow and modified coflow.
Laplacian flow and coflow of a Locally Conformal Parallel $G_2$-structure

In [Manero-Otal-V.’17] we study the (LF) and (LcF) starting from an LCP $G_2$-structure.

Questions:
- There exists solution for these flows?
- The solutions remain LCP?
- Is there any correspondence between solutions?

We want to solve:

\[
\begin{align*}
\frac{d}{dt} \sigma_t &= \Delta_t \sigma_t, \\
\sigma_0 &= \sigma, \\
d\sigma_t &= 3 \tau_1(t) \wedge \sigma_t, \\
d * t \sigma_t &= 4 \tau_1(t) \wedge * t \sigma_t.
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} \psi_t &= -\Delta_t \psi_t, \\
\psi_0 &= \psi, \\
d\psi_t &= 4 \tau_1(t) \wedge \psi_t, \\
d * t \psi_t &= 3 \tau_1(t) \wedge * t \psi_t.
\end{align*}
\]

Let us study them independently for a particular ansatz.
Our ansatz

Suppose that \( \{e^1, \ldots, e^7\} \) is an orthonormal local coframe in a \( G_2 \)-manifold \( (M^7, \sigma) \).

**Defomation:** Consider a time-dependent coframe \( \{x^1(t), \ldots, x^7(t)\} \)

\[ x^k(t) = h_k(t)e^k, \]

with \( h_k(t) \) differentiable functions, \( h_k(t) \neq 0 \) and \( h_k(0) = 1 \).

**Notation:** \( x^k \equiv x^k(t) \).

We define a one-parameter family of \( G_2 \)-structures on \( M \) as:

\[ \sigma_t = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}, \]

\[ \psi_t = x^{3456} + x^{1256} + x^{1234} - x^{2467} + x^{2357} + x^{1457} + x^{1367}. \]

In terms of the basis \( \{e^1, \ldots, e^7\} \):

\[ \sigma_t = h_{127}e^{127} + h_{347}e^{347} + h_{567}e^{567} + h_{135}e^{135} \]
\[ - h_{146}e^{146} - h_{236}e^{236} - h_{245}e^{245}, \]

\[ \psi_t = h_{3456}e^{3456} + h_{1256}e^{1256} + h_{1234}e^{1234} - h_{2467}e^{2467} \]
\[ + h_{2357}e^{2357} + h_{1457}e^{1457} + h_{1367}e^{1367}, \]

where \( h_{ijk} \) stands for the product \( h_i(t)h_j(t)h_k(t) \).
LCP flow: Solving $\frac{d}{dt}\sigma_t = \Delta_t\sigma_t$

Our ansatz:

- $x^k = h_k(t)e^k$, ($h_k(t)$ are the unknowns!!)
- $\sigma_t = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}$.
- $\{e^1, \ldots, e^7\}$ is orthonormal.

Direct computations:

$$\frac{d}{dt}\sigma_t = \sum_{(i,j,k) \in I} \left( \frac{h_i'}{h_i} + \frac{h_j'}{h_j} + \frac{h_k'}{h_k} \right) x^{ijk} - \sum_{(i,j,k) \in J} \left( \frac{h_i'}{h_i} + \frac{h_j'}{h_j} + \frac{h_k'}{h_k} \right) x^{ijk},$$

where $I = \{(127), (135), (347), (567)\}$ and $J = \{(146), (236), (245)\}$.

- Now, $\sigma_t$ solves the evolution equation for the 3-form, if and only if $\Delta$ has the following expression:

$$\Delta_t\sigma_t = \sum_{(i,j,k) \in I} \Delta_{ijk} x^{ijk} - \sum_{(i,j,k) \in J} \Delta_{ijk} x^{ijk},$$

where

$$\Delta_{ijk} = \frac{h_i'}{h_i} + \frac{h_j'}{h_j} + \frac{h_k'}{h_k}, \quad (i,j,k) \in I \cup J.$$

- Moreover:

$$\Delta_{abc} = \Delta_{pqr} \Rightarrow h_a h_b h_c = h_p h_q h_r.$$
Example

Consider a solvmanifold (compact quotient of solvable Lie group by lattice of maximal rank, $M = G/\Gamma$) whose Lie algebra is defined by:

▶ **Structure equations** in terms of basis $\{e^1, \ldots, e^7\}$:

$$c_p_1 = (-e^{17}, -e^{27}, -e^{37}, -e^{47}, -e^{57}, -e^{67}, 0) \quad (\leadsto de^1 = -e^1 \wedge e^7)$$

▶ *(invariant)* LCP $G_2$-structure:

$$\sigma_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$ 

▶ In terms of basis $x^k = h_k(t)e^k$:

$$c_p_1 = \left(-\frac{1}{h_7}x^{17}, -\frac{1}{h_7}x^{27}, -\frac{1}{h_7}x^{37}, -\frac{1}{h_7}x^{47}, -\frac{1}{h_7}x^{57}, -\frac{1}{h_7}x^{67}, 0\right).$$

▶ **Family of $G_2$-structures**: (We do not know if they are LCP!!)

$$\sigma_t = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}.$$ 

▶ **Laplacian**:

$$\Delta_t \sigma_t = -\frac{1}{h_7^2} \left[ 8 \left(x^{127} + x^{347} + x^{567}\right) + 9 \left(x^{135} - x^{146} - x^{236} - x^{245}\right) \right].$$
Example

Laplacian:

\[
\Delta_t \sigma_t = - \frac{1}{h_7^2} \left[ 8 \left( x^{127} + x^{347} + x^{567} \right) + 9 \left( x^{135} - x^{146} - x^{236} - x^{245} \right) \right].
\]

Laplacian:

\[
\Delta_t \sigma_t = - \frac{1}{h_7^2} \left[ 8 \left( x^{127} + x^{347} + x^{567} \right) + 9 \left( x^{135} - x^{146} - x^{236} - x^{245} \right) \right].
\]

Thus:

\[
\begin{align*}
\Delta_{127} &= \Delta_{347} = \Delta_{567} = -\frac{8}{h_7^2} \implies h_1 h_2 h_7 = h_3 h_4 h_7 = h_5 h_6 h_7. \\
\Delta_{135} &= \Delta_{146} = \Delta_{236} = \Delta_{245} = -\frac{9}{h_7^2} \implies h_1 h_3 h_5 = h_1 h_4 h_6 = h_2 h_3 h_6 = h_2 h_4 h_5.
\end{align*}
\]

Solving the blue system: \( h_1 = h_2 = h_3 = h_4 = h_5 = h_6 = h(t) \).

The evolution equation is equivalent to the system...
Example

The evolution equation is equivalent to the system

\[
\begin{align*}
 \frac{-2}{h_7^2} &= \frac{h'_7}{h_7} \\
 \frac{-3}{h_7^2} &= \frac{h'_7}{h}
\end{align*}
\]

Solution: \( h(t) = (1 - 4t)^{3/4} \) and \( h_7(t) = (1 - 4t)^{1/2} \).

Conclusion:

\[
\sigma_t = (1 - 4t)^2(e^{127} + e^{347} + e^{567}) + (1 - 4t)^{9/4}(e^{135} - e^{146} - e^{236} - e^{245})
\]

for \( t \in (-\infty, \frac{1}{4}) \) solves the evolution equation \( \frac{d}{dt} \sigma_t = \Delta_t \sigma_t \).

Moreover, can be checked that it remains LCP for any \( t \):

\[
\begin{align*}
 d\sigma_t &= 3\tau_1(t) \wedge \sigma_t, \\
 d \ast \sigma_t &= 4\tau_1(t) \wedge (\ast \sigma_t),
\end{align*}
\]

with \( \tau_1(t) = e^7 \).

Therefore, it is a solution for the LCP Laplacian flow.

Finally, observe that the metric \( g_t \) remains Einstein for all \( t \in (-\infty, \frac{1}{4}) \) since

\[
Ric(g_t) = -\frac{6}{1 - 4t} g_t.
\]
**LCP coflow: Solving** \[ \frac{d}{dt} \psi_t = -\Delta_t \psi_t \]

Similarly as the LCP flow, using \( x^k = h_k(t)e^k \), and taking into account that

\[
\psi_t = x^{3456} + x^{1256} + x^{1234} - x^{2467} + x^{2357} + x^{1457} + x^{1367}.
\]

\[
\frac{d}{dt} \psi_t = \sum_{(l,m,n,o)\in K} \left( \frac{h'_l}{h_l} + \frac{h'_m}{h_m} + \frac{h'_n}{h_n} + \frac{h'_o}{h_o} \right) x^{lmno} - \left( \frac{h'_2}{h_2} + \frac{h'_4}{h_4} + \frac{h'_6}{h_6} + \frac{h'_7}{h_7} \right) x^{2467},
\]

where \( K = \{(1234), (1256), (1367), (1457), (2357), (3456)\} \).

- Now, \( \sigma_t \) solves the evolution equation for the 4-form, if and only if \( \Delta \) has the following expression:

\[
\Delta_t \psi_t = \sum_{(l,m,n,o)\in K} \Delta_{lmno} x^{lmno} - \Delta_{2467} x^{2467},
\]

where

\[
\Delta_{lmno} = \frac{h'_l}{h_l} + \frac{h'_m}{h_m} + \frac{h'_n}{h_n} + \frac{h'_o}{h_o}, \quad (l, m, n, o) \in K \cup \{2467\}.
\]

- Moreover:

\[
\Delta_{lmno} = \Delta_{pqrs} \Rightarrow h_l h_m h_n h_o = h_p h_q h_r h_s.
\]
Solutions of the Laplacian flow and coflow of an LCP $G_2$-structure

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Motivation and $G_2$ background

Laplacian flow and coflow

Laplacian flow and coflow of a LCP $G_2$-structure

Results

Solvmanifolds with an LCP $G_2$-structure

[Chiossi-Fino’06] Obtained a family of solvmanifolds endowed with an LCP $G_2$-structure as a rank one solvable extension of 6-dim nilpotent Lie groups endowed with SU(3)-structure.

.cp\_1 = (-e^{17}, -e^{27}, -e^{37}, -e^{47}, -e^{57}, -e^{67}, 0);

.cp\_2 = \left( -\frac{4}{3} e^{17} + \frac{2}{3} e^{36}, -e^{27}, -\frac{2}{3} e^{37}, -e^{47}, -e^{57}, -\frac{2}{3} e^{67}, 0 \right);

cp\_3 = \left( -\frac{3}{2} e^{17} + \frac{1}{2} e^{36}, e^{45}, -\frac{3}{4} e^{27}, -\frac{3}{4} e^{37}, -\frac{3}{4} e^{47}, -\frac{3}{4} e^{57}, -\frac{3}{4} e^{67}, 0 \right);

cp\_4 = \left( -\frac{7}{5} e^{17} + \frac{2}{5} e^{36}, \frac{2}{5} e^{45}, -\frac{6}{5} e^{27} - \frac{2}{5} e^{46}, \frac{4}{5} e^{37}, -\frac{4}{5} e^{47}, -\frac{4}{5} e^{57}, -\frac{4}{5} e^{67}, 0 \right);

cp\_5 = \left( -\frac{5}{4} e^{17} + \frac{1}{2} e^{45}, -\frac{5}{4} e^{27} - \frac{1}{2} e^{46}, \frac{1}{2} e^{37}, -\frac{1}{2} e^{47}, -\frac{1}{2} e^{57}, -\frac{1}{2} e^{67}, 0 \right);

cp\_6 = \left( -\frac{4}{3} e^{17} + \frac{1}{3} e^{36}, \frac{1}{3} e^{45}, -\frac{4}{3} e^{27} + \frac{1}{3} e^{35}, -\frac{1}{3} e^{46}, -\frac{2}{3} e^{37}, -\frac{2}{3} e^{47}, -\frac{2}{3} e^{57}, -\frac{2}{3} e^{67}, 0 \right);

cp\_7 = \left( -\frac{6}{5} e^{17} + \frac{2}{5} e^{36}, -\frac{3}{5} e^{27}, -\frac{3}{5} e^{37}, \frac{2}{3} e^{26} - \frac{6}{5} e^{47}, \frac{2}{5} e^{35} - \frac{6}{5} e^{57}, -\frac{3}{5} e^{67}, 0 \right);

**Main result:** Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel $G_2$-structure admits

- ▶ a long time LCP solution to the Laplacian flow.
- ▶ a long time LCP solution to the Laplacian coflow.
More solutions to the LCP Laplacian flow & coflow

For the rest of the cases \( c p_i \) with \( i = \{2, \ldots, 7\} \), we consider a basis \( \{x^1, \ldots, x^7\} \) of 1-forms given by \( x^k = h_k(t) e^k \), and a particular type of functions \( h_k(t) \), inspired by the previous example:

\[
\begin{align*}
\text{flow} &\Rightarrow h_k(t) = (1 - \alpha t)^{\beta_k}, \\
\text{coflow} &\Rightarrow h_k(t) = (1 - \gamma t)^{\delta_k}.
\end{align*}
\]

<table>
<thead>
<tr>
<th>( c p_i )</th>
<th>( \alpha )</th>
<th>( (\beta_1, \ldots, \beta_7) )</th>
<th>( \gamma )</th>
<th>( (\delta_1, \ldots, \delta_7) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c p_1 )</td>
<td>4</td>
<td>( \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, 1\right) )</td>
<td>-6</td>
<td>( \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1\right) )</td>
</tr>
<tr>
<td>( c p_2 )</td>
<td>( \frac{10}{3} )</td>
<td>( \left(\frac{9}{10}, \frac{4}{5}, \frac{7}{10}, \frac{4}{5}, \frac{7}{10}, \frac{1}{2}\right) )</td>
<td>( \frac{-16}{3} )</td>
<td>( \left(\frac{1}{4}, \frac{5}{16}, \frac{5}{8}, \frac{5}{16}, \frac{3}{8}, \frac{1}{2}\right) )</td>
</tr>
<tr>
<td>( c p_3 )</td>
<td>3</td>
<td>( \left(1, \frac{5}{6}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, 1\right) )</td>
<td>-5</td>
<td>( \left(\frac{1}{5}, \frac{3}{10}, \frac{7}{20}, \frac{7}{20}, \frac{7}{20}, \frac{7}{20}, 1\right) )</td>
</tr>
<tr>
<td>( c p_4 )</td>
<td>( \frac{14}{5} )</td>
<td>( \left(1, \frac{13}{14}, \frac{11}{14}, \frac{5}{7}, \frac{11}{14}, \frac{5}{7}, 1\right) )</td>
<td>( \frac{-24}{5} )</td>
<td>( \left(\frac{5}{24}, \frac{1}{4}, 1, \frac{3}{8}, \frac{1}{3}, 1, 1\right) )</td>
</tr>
<tr>
<td>( c p_5 )</td>
<td>3</td>
<td>( \left(\frac{11}{12}, \frac{11}{12}, \frac{5}{6}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, 1\right) )</td>
<td>-5</td>
<td>( \left(\frac{1}{4}, \frac{1}{4}, \frac{3}{10}, \frac{2}{5}, \frac{7}{20}, \frac{7}{20}, 1\right) )</td>
</tr>
<tr>
<td>( c p_6 )</td>
<td>( \frac{8}{3} )</td>
<td>( \left(1, 1, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, 1\right) )</td>
<td>( \frac{-14}{3} )</td>
<td>( \left(\frac{3}{14}, \frac{3}{14}, \frac{5}{14}, \frac{5}{14}, \frac{5}{14}, \frac{5}{14}, 1\right) )</td>
</tr>
<tr>
<td>( c p_7 )</td>
<td>( \frac{14}{5} )</td>
<td>( \left(\frac{13}{14}, \frac{10}{14}, \frac{10}{14}, \frac{13}{14}, \frac{13}{14}, \frac{10}{14}, 1\right) )</td>
<td>( \frac{-24}{5} )</td>
<td>( \left(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, 1, \frac{3}{8}, \frac{1}{2}\right) )</td>
</tr>
</tbody>
</table>
Relation Theorem

The founded solutions for the LCP flow and the LCP coflow are related, as the following Theorem shows:

**Theorem**

Let $\sigma_t$ and $\tilde{\sigma}_t$ be two different families of $G_2$ structures on $\mathfrak{cp}_i$ with $i = 1, \ldots, 7$, where

$$h_k(t) = (1 - \alpha t)^{\beta_k}, \quad \beta_7 = \frac{1}{2}, \quad \text{and} \quad \tilde{h}_k(t) = (1 - \gamma t)^{\delta_k}, \quad \delta_7 = \frac{1}{2}.$$

If the defining parameters of the functions $h_i(t)$ and $\tilde{h}_i(t)$ are related by:

$$\gamma = \alpha \left( \frac{2 - \sum_{j=1}^{7} \beta_j}{2} \right), \quad \delta_k = \frac{1}{2} + \frac{1 - 2 \beta_k}{-2 + \sum_{j=1}^{7} \beta_j} \quad \text{for} \quad k \in \{1, \ldots, 7\}.$$

Then:

(i) $\sigma_t$ is LCP if and only if $\tilde{\sigma}_t$ is LCP.

(ii) $\sigma_t$ solves the Laplacian flow if and only if $\tilde{\psi}_t = \ast \tilde{\sigma} \tilde{\sigma}_t$ solves the Laplacian coflow.
Open problems

We have obtained some long-time solutions to the LCP flow and coflow.

Open problems:
- Study short time-existence and uniqueness of solution.
- Study the behavior of limit of solutions. Are they parallel structures?

Thank you!!