Bulk-boundary in higher dimensions:

2D
Different topological phases in 2D

2D Topological insulator
The (integer) Quantum Hall Effect

- Electrons confined in 2D under strong perpendicular magnetic field
- Time reversal symmetry is broken

- Bulk: cyclotron orbits → **Landau levels**
- Boundary: skipping orbits → metallic edge channels
- Robust against disorder and interactions
The (integer) Quantum Hall Effect

Quantized Hall conductivity:

\[ J_y = \sigma_{xy} E_x \]
\[ \sigma_{xy} = \frac{N e^2}{h} \]

Integer accurate to \(10^{-9}\)

von Klitzing, Dorda, Pepper, PRL 1980
The Z topological invariant

Thouless, Kohmoto, Nightingagle, den Nijs (1982):

Number of topologically protected edge channels $\rightarrow N \equiv n$ → Topological invariant of the LL band structure

Berry phase: line integral of $A_m = i \langle u_m | \nabla_k | u_m \rangle$ → Berry flux $F_m = \nabla \times A_m$

The Chern invariant is the total Berry flux in the Brillouin zone

$n_m = \frac{1}{2\pi} \int d^2k F_m$ → $n = \sum_{m=1}^{N} n_m$

Bulk-boundary correspondence:

$N_R - N_L = \Delta n$

Chiral edge states
Quantum Spin Hall Effect

- Spin-orbit interaction allows for a different topological class of insulating band structures
- Time Reversal Symmetry is conserved

Hall conductivity = 0, but spin Hall conductivity ≠ 0:

\[ J_{y}^{\uparrow} - J_{y}^{\downarrow} = \sigma_{xy} E_x \]

\[ \sigma_{xy} = \frac{e^2}{h} \]

Gapless spin-filtered edge states that are robust against non-magnetic disorder and interactions thanks to T-symmetry

Edge states form a unique 1D conductor that is essentially half of an ordinary 1D conductor
The $\mathbb{Z}_2$ topological invariant

Number of topologically protected edge channels is 0 or 1

If perpendicular spin $S_z$ is conserved

The up and down spins have independent “spin” Chern numbers $n^\uparrow$, $n^\downarrow$

$$n^\uparrow + n^\downarrow = 0$$

but the difference defines a quantized spin Hall conductivity

$$n_\sigma = (n^\uparrow - n^\downarrow)/2$$

The $\mathbb{Z}_2$ invariant is the parity of the spin Chern number

$$\nu = n_\sigma \mod 2$$

Quantum Spin Hall (TRI topological insulator)

spin-momentum locking

Helical edge states

M. Z. Hasan and C. L. Kane, review (2010)
The $Z_2$ topological invariant

When $S_z$ is not conserved?

Three approaches to calculate $Z_2$:

- Exploit adiabatic continuity to a Hamiltonian which has extra symmetry
- Evaluate the $Z_2$ invariant directly with the knowledge of the Bloch wave functions for the occupied energy bands (à la Chern number)
- Characterize the Pfaffian function in the high symmetry points of the BZ

A two-dimensional cylinder threaded by magnetic flux $\Phi$. When the cylinder has a circumference of a single lattice constant, $\Phi$ plays the role of the edge crystal momentum $k_x$ in band theory.

The time-reversal invariant fluxes $\Phi=0$ and $h/2e$ correspond to edge time-reversal invariant momenta $\Lambda_1$ and $\Lambda_2$. $\Lambda_a$ are projections of pairs of the four bulk time-reversal momenta $\Gamma_{i\nu}$, which reside in the two-dimensional Brillouin zone.

\[
\Theta = \exp(i\pi S_y)K
\]

Time symmetry operator, $S_y$ spin, $K$ complex conjugation

\[
\begin{align*}
\theta_{mn}(k) & = \langle \mu_{n-k}|\Theta|\mu_{nk} \rangle \\
\det[w] & = \text{Pf}[w]^2
\end{align*}
\]

\[
\delta_i = \frac{\sqrt{\det[w(\Gamma_i)]}}{\text{Pf}[w(\Gamma_i)]} = \pm 1
\]

The $Z_2$ invariant $\nu$ is:

\[
(-1)^\nu = \prod_{i=1}^{4} \delta_i
\]

Graphene as a QSH system

**Kane-Mele model:** Graphene + intrinsic spin-orbit

\[ \mathcal{H}_0 = -i\hbar v_F \psi^\dagger (\sigma_x \tau_z \partial_x + \sigma_y \partial_y) \psi. \]

\[ \mathcal{H}_{SO} = \Delta_{SO} \psi^\dagger \sigma_z \tau_z \sigma_z \psi. \]

...but real graphene has negligible SOC

\[ \Delta_{SO} = 10^{-3}\text{meV} \]

**Proposal for engineering SOC in graphene**

Heavy adatom doping or placing it on substrate materials with strong SOC

Exploration of other low-buckled graphene-like materials with stronger SOC, such as silicene, germanene, stanene.

In this line a gap is opened in a 2D Dirac semimetal, but there is an alternative route...

...by inducing a band inversion in narrow gap semiconductors
2D Topological insulator

- HgTe/CdHgTe quantum wells

Bernevig, Hughes, and Zhang (BHZ) model

\[ \Delta = 10 \text{meV} \]

Molenkamp’s group
Bulk-boundary in higher dimensions:

3D

- First generation experiments: semiconducting alloy Bi$_{1-x}$Sb$_x$
- Second generation experiments: Bi$_2$Se$_3$, Bi$_2$Te$_3$ and Sb$_2$Te$_3$
Topological superconductors
Topological superconductors

- Gapped bulk → superconducting gap
- Protected edge states → called **Majorana** states

\[ \gamma_L \quad \gamma_R \]

**Topological superconductor**

- We need certain requirements to build a TS:
  - Spin degeneracy needs to be broken: spinless liquid
  - Non conventional SC pairing: p-wave type

**Unpaired Majorana fermions in quantum wires**

A Yu Kitaev
© 2001 Uspekhi Fizicheskikh Nauk, Russian Academy of Sciences
Physics-Uspekhi, Volume 44, Supplement
Superconductivity and topology

Bogoliubov-de Gennes Hamiltonian for spinless electrons

\[ H - \mu N = \frac{1}{2} \sum_k \begin{pmatrix} c_k & c_{-k} \end{pmatrix} H_{\text{BdG}}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \]

\[ H_{\text{BdG}}(k) = \begin{pmatrix} H_0(k) & \Delta(k) \\ \Delta^\dagger(k) & -H_0^*(\bar{k}) \end{pmatrix} \]

\[ \Delta \] is the BCS mean field pairing potential

\[ H_{\text{BdG}} \] has particle-hole symmetry:

\[ \Lambda H_{\text{BdG}}(k) \Lambda^{-1} = -H_{\text{BdG}}(-k) \]

Particle-hole operator:

\[ \Lambda = K \tau_x \]

For spinless e’s odd parity SC:

\[ \Delta(-k) = -\Delta(k) \]

1D: p-wave SC

\[ \Delta(k) \sim \Delta_0 k \]

2D: px+ipy chiral SC

\[ \Delta(k) \sim \Delta_0 (k_x + ik_y) \]
Superconductivity and topology in 1D

Diagonalizing the BdG Hamiltonian: Bogoliubov quasiparticles $\Gamma$’s

At $E=0$:

$$\Gamma_0 = \frac{\gamma_L + i\gamma_R}{\sqrt{2}}$$
$$\Gamma_0^\dagger = \frac{\gamma_L - i\gamma_R}{\sqrt{2}}$$

$\gamma$’s are self-conjugated, the defining feature of Majorana fermion

$\gamma_{L,R}^\dagger = \gamma_{L,R}$

A single unpaired bound state at $E=0$ is topological because it cannot move away from $E=0$

In the 1D Kitaev model, the presence or absence of such a zero mode is determined by the $\mathbb{Z}_2$ topological invariant
Majorana bound states in 1DTS

Emergent quasiparticles bound to boundaries or topological defects in superconductors at zero energy (midgap excitations)

Each Majorana is its own antiparticle: \( \gamma^\dagger = \gamma \)

Quantum superposition of a particle and a hole, with the same weight (no net charge, no net spin)

In finite size systems they always come in pairs. When they approach each other, they annihilate

Upon exchange they follow Ising anion statistics

\( \gamma = c + c^\dagger \)
A well separated pair defines a degenerate two-level system (empty or occupied fermionic state) → a **qubit**

\[ |0\rangle, \quad |1\rangle = f^{\dagger}|0\rangle \quad f^{\dagger} = \frac{1}{2}(\gamma_1 - i\gamma_2) \]

**Key property:** the quantum information in this qubit is stored **non locally**

The state cannot be perturbed with a local measurement on one of the bound states or local noise

They could be used as building blocks of **topological quantum computation**
Building a 1D Topological Superconductor

Fu-Kane proposal I

Helical half-metal

+ proximity to s-wave superconductor

Topological p-wave Superconductor

1D

Helical half-metal

\( k_x \)

\( \epsilon \)

Building a 1D Topological Superconductor

Fu-Kane proposal I

Helical half-metal

+ proximity to s-wave superconductor

Topological p-wave Superconductor

1D

s-wave pairing

Helical half-metal

topological gap

Building a 1D Topological Superconductor

Fu-Kane proposal I

1D

Helical half-metal
+
proximity to s-wave superconductor

Topological p-wave Superconductor

(2D)

Quantum Spin Hall
Building a 1D Topological Superconductor

Fu-Kane proposal I

1D

Helical half-metal

+ proximity to s-wave superconductor

Topological p-wave Superconductor

(2D)

s-wave SC

Quantum Spin Hall

Ferromagnetic insulator

Building a 1D Topological Superconductor

Fu-Kane proposal I

Helical half-metal

+ proximity to s-wave superconductor

Topological p-wave Superconductor

1D

(2D)

s-wave SC
(non-trivially) gapped

Quantum Spin Hall

Ferromagnetic insulator

γ1

( trivially ) gapped

γ2

Promising signatures of Majoranas in Nb/HgTe (SC/TI)

Figure 1 | Geometry of the Josephson junction and predicted Andreev spectrum.

Figure 3 | 2D plots of the bin counts and Shapiro step amplitudes for the

Missing Shapiro steps:

4π Josephson effect?
Building a 1D Topological Superconductor

Oreg-Lutchyn proposal $\rightarrow$ Majorana Nanowire

$B > B_c$

Majorana modes

$\sqrt{1}$ Topological superconductor $\sqrt{2}$

$S$–wave superconductor

Majorana Nanowire

Semiconducting nanowire with strong SO (Rashba) coupling

- Zeeman field perpendicular to SO term $\rightarrow$ spinless helical region
- Induced s-wave superconductivity

Topological regime for $B > B_c$

$B_c = \sqrt{\Delta^2 + \mu^2}$

Effective p-wave pairing

$\Delta_{--} = \frac{i\alpha p_x \Delta}{\sqrt{\alpha^2 p_x^2 + B^2}}$

$\frac{\epsilon}{\Delta}$ vs $p_x$

$B/B_c = 0.00$
Experimental realisation: nanowires

Signatures of Majorana Fermions in Hybrid Superconductor-Semiconductor Nanowire Devices

V. Mourik,†‡ K. Zuo,†‡ S. M. Frolov,‡ S. R. Plissard,‡ E. P. A. M. Bakkers,†‡ L. P. Kouwenhoven†

Leo Kouwenhoven's lab

Experimental realisation: nanowires

InAs/Al semiconductor/superconductor NW epitaxy

Charlie Marcus' lab

\[ \frac{dI}{dV} (e^2/h) \]
Experimental realisation: atom chains

Pairing + Spin-orbit from Pb substrate
Zeeman replaced by ferromagnetism of Fe atom chain

Ali Yazdani’s lab

Experimental realisation: epitaxial 2DEGs

Good proximity effect in 2DEGs is now possible

Charlie Marcus' lab

Majorana bound state

1D TS

200 nm
Topological superconductors in 2D

- Chiral $px + ipy$ SC
### Periodic table of topological systems

- **10 symmetry classes of single-particle fermionic Hamiltonians**
  - Symmetries (TRS, PHS, SLS)
  - Dimensionality $d$
  - Topological invariants

<table>
<thead>
<tr>
<th></th>
<th>TRS</th>
<th>PHS</th>
<th>SLS</th>
<th>$d=1$</th>
<th>$d=2$</th>
<th>$d=3$</th>
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<td>Standard (Wigner-Dyson)</td>
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<td>A (unitary)</td>
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<td>0</td>
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<td>$\mathbb{Z}$</td>
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<td>-</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>-</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
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<tr>
<td>Chiral (sublattice)</td>
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<tr>
<td>AIII (chiral unitary)</td>
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<td>0</td>
<td>1</td>
<td>$\mathbb{Z}$</td>
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<td>$\mathbb{Z}$</td>
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<tr>
<td>BDI (chiral orthogonal)</td>
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<td>+1</td>
<td>1</td>
<td>$\mathbb{Z}$</td>
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<td>-</td>
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<td>-</td>
<td>-</td>
<td>$\mathbb{Z}$</td>
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</tbody>
</table>

- **QHE**
- **2DTI**
- **3DTI**
- **1D TS: Kitaev model, Majorana NW**
- **Realistic Majorana NW**
- **2DTS: SC + QAH, spinless px+ipy SC**
- **$^3$HeB**
- **2DTS: SC + TI surface**
Applications of topological matter

- TI: information highways
  - Non-dissipative edge currents
    - No superconductivity or magnetic field required

- TI: spintronics
  - Pure spin currents along edge states

- TI: Topological electromagnetic response
  - Effective monopoles, Topological Casimir effect, topological magnetoelectric effect, field confinement, etc…

- TS: Majoranas for topological quantum computation
  - Qubits topologically protected against decoherence
  - Non-abelian braiding operations
International efforts

- EU initiative
  - 1000 M€ investment in quantum technologies
  - Topological superconductors, a key to protect ‘quantumness'
  - Final goal: universal quantum computation

Quantum Technologies Timeline

Thank you for your attention!