Discrete geometry of polygons and Hamiltonian structures

Gloria Marí Beffa
University of Wisconsin - Madison

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Discrete geometric realizations

1. Assume we are in $\mathbb{RP}^1$ and let $\{V_n\}$ be a lift of a projective polygon to $\mathbb{R}^2$ such that $\det(V_n, V_n+1) = 1$ for all $n$.

Let $p_n = \det(V_n-1, V_n+1)$ be a polygonal curvature and $q_n = \frac{1}{p_n(p_n+1)}$ the cross ratio of the projective points with lifts $V_n-1, V_n, V_n+1, V_n+2$.

Then the projective tangential flow $\{V_n\}^t = \frac{1}{p_n(V_n+1-V_n-1)}$ induces the evolution $q_n^t = q_n(q_n+1-q_n-1)$, the Volterra model.

We say that the projective tangential flow is a projective realization of the Volterra model.
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Discrete geometric realizations

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be a polygonal curvature and $q_n = \frac{1}{p_{n+1}p_n}$ the cross ratio of the projective points with lifts $V_{n-1}, V_n, V_{n+1}, V_{n+2}$. Then the projective tangential flow

$$\begin{aligned}
(V_n)_t &= \frac{1}{p_n} (V_{n+1} - V_{n-1})
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induces the evolution

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the Volterra model. We say that the projective tangential flow is a projective realization of the Volterra model.
2. If $V_n \in \mathbb{R}^2$ is a polygon in equicentro-affine plane ($\text{SL}(2, \mathbb{R})$ acting linearly), with invariants

$$a_n = \det(V_n, V_{n+1}), \quad \kappa_n = \frac{\det(V_n, V_{n+2})}{\det(V_{n+1}, V_{n+2})}$$
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then, whenever

$$(V_n)_t = \frac{a_{n-1}}{a_n} V_{n+1}$$

and $p_n = \frac{a_n}{a_{n+1}}$, $q_n = \kappa_n$, we have

$$(p_n)_t = p_n (q_n - q_{n+1}),$$

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which is the Toda Lattice.
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Assume \( u : J \subset \mathbb{R}^2 \to \mathbb{R}^3 \) is a solution of the Vortex filament flow equation (Localized induction flow)

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    u_t = \kappa B
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Assume $u : J \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a solution of the Vortex filament flow equation (Localized induction flow)

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where $\kappa$ is the curvature of the flow $u$ and $B$ is the binormal. Then, curvature and torsion of the flow satisfy an equation equivalent to the Nonlinear Schrödinger equation (NLS). If

$$\phi = \kappa e^{i \int \tau dx}, \quad \phi_t = i \phi_{xx} + \frac{i}{2} |\phi|^2 \phi$$

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We say the Vortex Filament flow is an Euclidean realization of NLS. Or NLS is the Vortex filament flow defined on the moduli space of Euclidean curves.
The moduli space of twisted polygons

Let $M = G/H$ with $G$ semisimple Lie group acting naturally on $M$. We say a polygon $\{u_n\} \in M^\infty$ is twisted with period $N$ if there exists $g \in G$ such that $u_{N+n} = g \cdot u_n$, for all $n$. The element $g$ is called the monodromy of the polygon.

How can we find coordinates for the moduli space of twisted polygons?

We define a right (resp. left) discrete moving frame associated to $\{u_n\}$ as an equivariant map $\rho: U \subset M^N \rightarrow G^N$ wrt the diagonal action on $M^N$ and the right inverse (resp. left) action on $G^N$. If $\rho(u) = (\rho_n)$, we say $\rho_n$ is the moving frame at the vertex $u_n$.

Joint work with J. P. Wang, “Hamiltonian structures and integrable evolutions of twisted polygons in $\mathbb{RP}^n$”, (2013), and A. Calini “Integrable evolutions of twisted polygons in centro-affine $\mathbb{R}^m$” (in progress).
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We define the right (resp. left) \textit{discrete Serret–Frenet equations} to be

\[ \rho_{n+1} = K_n \rho_n \quad (\text{resp.} \quad \rho_{n+1} = \rho_n K_n). \]
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The group elements \( \{K_n\}_{n=1}^N \) functionally generate all invariants of the polygon under the diagonal action (Mansfield, MB, Wang 2013).
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\( \{K_n\}_{n=1}^N \) define (local) coordinates in the moduli space of polygons under the action of \( G \).
Example
Let \{u_n\} be a twisted polygon in \(\mathbb{RP}^m\).
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The map

\[
\rho : (\mathbb{RP}^m)^N \rightarrow \text{SL}(m + 1, \mathbb{R})^N
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\rho(\{u_n\}) = \{(V_{n+m}, \ldots, V_{n+1}, V_n)\}
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Let \( \{u_n\} \) be a twisted polygon in \( \mathbb{R}P^m \). One can prove that if \( \{u_n\} \) is non degenerate and if \( N \) and \( m + 1 \) are coprime, there is a unique lift to a polygon in \( \mathbb{R}^{m+1} \), \( \{V_n\} \), such that \( \det(V_{n+m}, \ldots, V_{n+1} V_n) = 1 \) for all \( n \). The map

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\[
\rho_{n+1} = \rho_n \begin{pmatrix} k_n^m & 1 & 0 & \ldots & 0 \\ k_n^{m-1} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ k_n^1 & 0 & \ldots & 0 & 1 \\ (-1)^{m+1} & 0 & 0 & \ldots & 0 \end{pmatrix}
\]

where \( k_n^i \) are given by \( V_{n+m+1} = k_n^m V_{n+m} + \ldots k_n^1 V_{n+1} + (-1)^m V_n \) and \( \{k_n^i\} \) generate all other invariants of the polygon.
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In general

**Theorem**
Assume $M = G/H$. The moduli space of non degenerate twisted polygons in $M^N$ can be identified with an open subset of the quotient $G^N/H^N$, where $H^N$ acts on $G^N$ via the right gauge action

$$H^N \times G^N \to G^N$$

$$((h_n), (g_n)) \to (h_{n+1} g_n h_n^{-1})$$
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The question now is: is there a natural Poisson structure in $G^N$ such that right gauge is a Poisson map? If so we might be able to reduce it.
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The question now is: is there a natural Poisson structure in $G^N$ such that right gauge is a Poisson map? If so we might be able to reduce it.

They were classified by Semenov-Tian-Shansky in “Dressing transformations and Poisson Group actions”, (1985). We will describe the main bracket.
A Poisson structure on $G^N$

Assume $g$ to have an inner product $\langle \cdot, \cdot \rangle$ that identifies $g$ with $g^\ast$. Let $F : G^N \to \mathbb{R}$ be a function, we define the left gradient at $L = (L^n) \in G^N$ to be the element of $g^N$, $(\nabla_n F(L^n))$ such that, for any $(\xi^n) \in g^N$:

$$\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} F(\exp(\epsilon \xi^n)L^n) = \langle \nabla_n F(L^n), \xi^n \rangle.$$  

Analogously, the right gradient satisfies:

$$\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} F(\exp(\epsilon \xi^n)L^n) = \langle \nabla'_n F(L^n), \xi^n \rangle.$$  

Assume $g$ has a grading $g = g^+ \oplus h_0 \oplus g^-$, with $h_0$ commutative, $g^+$ dual to $g^-$. We define the classical $\mathbb{R}$-matrix to be map $R : g \to g$:

$$R(\xi^+ + \xi^-) = \frac{1}{2}(\xi^+ - \xi^-).$$  

Associated to $R$, there exists a 2-tensor $r$ such that:

$$r(\xi^\wedge \eta) = \langle \xi, R(\eta) \rangle,$$

$$r(\xi^\otimes \eta) = \langle \xi^+, \eta^- \rangle.$$
A Poisson structure on $G^N$

Assume $g$ to have an inner product $\langle \ , \ \rangle$ that identifies $g$ with $g^*$. 

Associated to $R$ there exists a 2-tensor $r$ such that $r(\xi \wedge \eta) = \langle \xi, R(\eta) \rangle$, $r(\xi \otimes \eta) = \langle \xi^+ , \eta^- \rangle$. 

Gloria Marí Beffa University of Wisconsin - Madison
Discrete geometry of polygons and Hamiltonian structures
A Poisson structure on $G^N$

Assume $\mathfrak{g}$ to have an inner product $\langle , \rangle$ that identifies $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $\mathcal{F} : G^N \to \mathbb{R}$ be a function, we define the left gradient at $L = (L_n) \in G^N$ to be the element of $\mathfrak{g}^N$, $(\nabla_n \mathcal{F}(L))$ such that, for any $(\xi_n) \in \mathfrak{g}^N$

$$
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \mathcal{F} \left( \exp(\epsilon \xi_n) L_n \right) = \langle \nabla_n \mathcal{F}(L), \xi_n \rangle.
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A Poisson structure on $G^N$

Assume $\mathfrak{g}$ to have an inner product $\langle \cdot, \cdot \rangle$ that identifies $\mathfrak{g}$ with $\mathfrak{g}^\ast$. Let $\mathcal{F} : G^N \to \mathbb{R}$ be a function, we define the left gradient at $L = (L_n) \in G^N$ to be the element of $\mathfrak{g}^N$, $(\nabla_n \mathcal{F}(L))$ such that, for any $(\xi_n) \in \mathfrak{g}^N$

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{F}(\exp(\epsilon \xi_n)L_n) = \langle \nabla_n \mathcal{F}(L), \xi_n \rangle.$$ 

Analogously, the right gradient satisfies

$$\frac{d}{d\epsilon}|_{\epsilon=0}\mathcal{F}(L_n \exp(\epsilon \xi_n)) = \langle \nabla'_n \mathcal{F}(L), \xi_n \rangle.$$
A Poisson structure on $G^N$

Assume $\mathfrak{g}$ to have an inner product $\langle , \rangle$ that identifies $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $\mathcal{F} : G^N \to \mathbb{R}$ be a function, we define the left gradient at $L = (L_n) \in G^N$ to be the element of $\mathfrak{g}^N$, $(\nabla_n \mathcal{F}(L))$ such that, for any $(\xi_n) \in \mathfrak{g}^N$

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \mathcal{F}(\exp(\epsilon \xi_n) L_n) = \langle \nabla_n \mathcal{F}(L), \xi_n \rangle.$$ 

Analogously, the right gradient satisfies

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \mathcal{F}(L_n \exp(\epsilon \xi_n)) = \langle \nabla'_n \mathcal{F}(L), \xi_n \rangle.$$ 

Assume $\mathfrak{g}$ has a grading $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_-$, with $\mathfrak{h}_0$ commutative, $\mathfrak{g}_+$ dual to $\mathfrak{g}_-$.
A Poisson structure on $G^N$

Assume $\mathfrak{g}$ to have an inner product $\langle , \rangle$ that identifies $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $\mathcal{F} : G^N \to \mathbb{R}$ be a function, we define the **left gradient** at $L = (L_n) \in G^N$ to be the element of $\mathfrak{g}^N$, $(\nabla_n \mathcal{F}(L))$ such that, for any $(\xi_n) \in \mathfrak{g}^N$

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{F}(\exp(\epsilon \xi_n) L_n) = \langle \nabla_n \mathcal{F}(L), \xi_n \rangle.$$  

Analogously, the **right gradient** satisfies

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{F}(L_n \exp(\epsilon \xi_n)) = \langle \nabla'_n \mathcal{F}(L), \xi_n \rangle.$$ 

Assume $\mathfrak{g}$ has a grading $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_-$, with $\mathfrak{h}_0$ commutative, $\mathfrak{g}_+$ dual to $\mathfrak{g}_-$. We define the **classical $R$-matrix** to be map $R : \mathfrak{g} \to \mathfrak{g}$

$$R(\xi_+ + \xi_h + \xi_-) = \frac{1}{2}(\xi_+ - \xi_-).$$
A Poisson structure on $G^N$

Assume $\mathfrak{g}$ to have an inner product $\langle , \rangle$ that identifies $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $\mathcal{F} : G^N \rightarrow \mathbb{R}$ be a function, we define the left gradient at $L = (L_n) \in G^N$ to be the element of $\mathfrak{g}^N$, $(\nabla_n \mathcal{F}(L))$ such that, for any $(\xi_n) \in \mathfrak{g}^N$

$$\frac{d}{d\epsilon}|_{\epsilon=0} \mathcal{F}(\exp(\epsilon \xi_n) L_n) = \langle \nabla_n \mathcal{F}(L), \xi_n \rangle.$$  

Analogously, the right gradient satisfies

$$\frac{d}{d\epsilon}|_{\epsilon=0} \mathcal{F}(L_n \exp(\epsilon \xi_n)) = \langle \nabla'_n \mathcal{F}(L), \xi_n \rangle.$$  

Assume $\mathfrak{g}$ has a grading $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_-$, with $\mathfrak{h}_0$ commutative, $\mathfrak{g}_+$ dual to $\mathfrak{g}_-$. We define the classical $R$-matrix to be map $R : \mathfrak{g} \rightarrow \mathfrak{g}$

$$R(\xi_+ + \xi_h + \xi_-) = \frac{1}{2}(\xi_+ - \xi_-).$$  

Associated to $R$ there exists a 2-tensor $r$ such that

$$r(\xi \wedge \eta) = \langle \xi, R(\eta) \rangle, \quad r(\xi \otimes \eta) = \langle \xi_+, \eta_- \rangle.$$
Define the **twisted Poisson bracket** in $G^N$ to be given by

$$\{\mathcal{F}, \mathcal{G}\}(L) = \sum_{s=1}^{N} r(\nabla_s \mathcal{F} \wedge \nabla_s \mathcal{G}) + \sum_{s=1}^{N} r(\nabla'_s \mathcal{F} \wedge \nabla'_s \mathcal{G})$$

$$- \sum_{s=1}^{N} r (\tau \otimes \text{id})(\nabla'_s \mathcal{F} \otimes \nabla_s \mathcal{G})) + \sum_{s=1}^{N} r (\tau \otimes \text{id})(\nabla'_s \mathcal{G} \otimes \nabla_s \mathcal{F})) .$$

The twisted Poisson bracket defines a Hamiltonian structure for which gauge action is a Poisson map.
A discrete geometric Poisson bracket

Assume $G$ has a Lie algebra $g$ with two gradations $g = g^+ \oplus h^0 \oplus g^-$ with $h^0$ commutative and $g^+$ and $g^-$ dual of each other, and $g = g_1 \oplus g_0 \oplus g_{-1}$, $g_1$ and $g_{-1}$ dual of each other.

Assume $M = G/H$ with $h = g_0 \oplus g_1$.

We say both gradations are compatible if $g_1 \subset g^+$, $g_{-1} \subset g^-$.

Theorem (MB 14) Assume $M = G/H$ and $g$ has two compatible gradations as above. The twisted Poisson structure defined on $G/N$, with $r$ associated to the classical $R$-matrix, is locally reducible to the quotient $G/N/H$.

Furthermore, any reduced Hamiltonian evolution with Hamiltonian functional $f$ is induced on the invariants by a local invariant polygonal vector field $X_f = (X_{f,n})$ in $M$. 
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Assume $G$ has a Lie algebra $\mathfrak{g}$ with two gradations $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_-$ with $\mathfrak{h}_0$ commutative and $\mathfrak{g}_+$ and $\mathfrak{g}_-$ dual of each other.
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$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_-$ with $\mathfrak{h}_0$ commutative and $\mathfrak{g}_+$ and $\mathfrak{g}_-$ dual of each other, and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ dual of each other. Assume $M = G/H$ with $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. 

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$$\mathfrak{g}_1 \subset \mathfrak{g}_+, \quad \mathfrak{g}_{-1} \subset \mathfrak{g}_-.$$
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**Theorem**

*(MB 14)* Assume $M = G/H$ and $\mathfrak{g}$ has two compatible gradations as above. The twisted Poisson structure defined on $G^N$, with $r$ associated to the classical $R$-matrix, is locally reducible to the quotient $G^N/H^N$. 

Gloria Marí Beffa  
University of Wisconsin - Madison  
Discrete geometry of polygons and Hamiltonian structures
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**Theorem**

(MB 14) Assume $M = G/H$ and $\mathfrak{g}$ has two compatible gradations as above. The twisted Poisson structure defined on $G^N$, with $r$ associated to the classical $R$-matrix, is locally reducible to the quotient $G^N/H^N$. Furthermore, any reduced Hamiltonian evolution with Hamiltonian functional $f$ is induced on the invariants by a local invariant polygonal vector field $X^f = (X^f_n)$ in $M$. 

Gloria Marí Beffa University of Wisconsin - Madison

Discrete geometry of polygons and Hamiltonian structures
Example

(MB, Wang 13) In the projective case $G = \text{PSL}(m+1)$ the gradations are:

1. $g^+ = \text{strictly lower triangular matrices}; h_0 = \text{diagonal matrices}; g^- = \text{strictly upper triangular matrices}.$

2. $g_1 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \ast & \cdots & \ast & 0 \end{pmatrix}, g_0 = \begin{pmatrix} \ast & \cdots & \ast & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \ast & 0 \end{pmatrix}, g_- = \begin{pmatrix} 0 & \cdots & 0 & \ast \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \ast & 0 \end{pmatrix}$

If we define $h = g_0 \oplus g_1$, then $\mathbb{P}^m = G/H$ is the standard description of the projective space as homogeneous space.

The two gradations are compatible and hence we have a reduced bracket.
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$$
\begin{align*}
g_1 &= \begin{pmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\ast & \cdots & \ast & 0
\end{pmatrix}, \\
h_0 &= \begin{pmatrix}
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\vdots & \ddots & \ddots & \vdots \\
\ast & \cdots & \ast & 0 \\
0 & \cdots & 0 & \ast
\end{pmatrix}, \\
g_{-1} &= \begin{pmatrix}
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$$\begin{pmatrix}
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\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0 \\
* & \ldots & * & 0
\end{pmatrix}, \begin{pmatrix}
* & \ldots & * & 0 \\
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\end{pmatrix}$$

If we define $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then $\mathbb{R}P^m = G/H$ is the standard description of the projective space as homogeneous space. The two gradations are compatible and hence we have a reduced bracket.
In the case \( n = 1 \) the bracket is given by: let \( f, g \) be functions of \( k_n \)

\[
\{ f, g \}(k_n) = \sum_n \frac{df}{dk_n} \left( \tau^{-1} - \tau + k_n(\tau - 1)(\tau + 1)^{-1}k_n \right) \frac{dg}{dk_n}
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which is one of two Hamiltonian structures for the modified Volterra lattice, the other one being \( \tau - \tau^{-1} \).
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Indeed, if a twisted polygon in \( \mathbb{RP}^1 \) has a lift \( \{V_n\} \) solution of

\[
(V_n)_t = f_n V_{n+1} + \alpha_n V_n,
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where \( f_n \) an arbitrary function of \( k_n \), and \( \alpha_n = -(1 + \tau)^{-1} k_{n-1} f_n \) is the unique choice that preserves \( \det(V_{n+1}, V_n) = 1 \),
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The vector field

$$X^f_n = \frac{\delta f}{\delta k_n} V_{n+1} + \alpha_n V_n$$

is the lifted vector field in $\mathbb{R}^2$. 

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Indeed, if a twisted polygon in $\mathbb{RP}^1$ has a lift $\{V_n\}$ solution of

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The vector field

$$X_n^f = \frac{\delta f}{\delta k_n} V_{n+1} + \alpha_n V_n$$

is the lifted vector field in $\mathbb{R}^2$. The choice $f = \sum_n \ln k_{n-1}$ defines an evolution equivalent to the modified Volterra chain.
Theorem

(MB, Wang 13) The evolution in $\mathbb{RP}^m$ whose unique lift to $\mathbb{R}^{m+1}$ is

$$(V_n)_t = \frac{1}{k_{n-1}^1} (V_{n+m} + k_n^m V_{n+m-1} + \cdots + k_n^2 V_{n+1}) + \alpha_n V_n$$

$\alpha_n$ uniquely determined by preservation of $\det(V_{n+m}, \ldots, V_n) = 1$, induces on $k_i^n$ (bi-Hamiltonian?) and completely integrable discretizations of $W_m$ algebras, (generalizations of the Boussinesq lattice), for any $m$. 
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The equations are given by

$$
\begin{align*}
  k_i^1 &= \frac{k_{i+1}^1}{k_i^1} - \frac{k_{i+1}^1}{k_{i-i}^1}, \quad i = 1, 2, \cdots, m - 1 \\
  k_m^m &= \frac{1}{k_1^1} - \frac{1}{k_{N-m}^1}
\end{align*}
$$

Under the Miura transformation $u^1 = \frac{1}{k^1_k \cdots k^1_1}$, $u^i = \frac{k_{i-1}^1}{k^1_k \cdots k_{i-1}^1}$, $i = 2, \cdots, N + 1$, they become

$$
\begin{align*}
  u^1_t &= -u^1(u^2_N - u^2_1) \\
  u^i_t &= u^{i+1}_t - u^i_{i-1} - u^i(u^2_{i-1} - u^2_1), \quad i = 2, 3 \cdots, m - 1 \\
  u^m_t &= u^1_r - u^1_{i-1} - u^m(u^2_{N-1} - u^2_1)
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$$
Theorem

(MB, Wang 13) The right bracket for the parabolic $r$ tensor

$$\{F, G\}_{\text{right}}(L) = \sum_n r(\nabla'_n F(L) \wedge \nabla'_n G(L))$$

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They were proved to be compatible in recent work with Calini. The study of the compatibility of the Poisson pair is based on lifting the two Hamiltonian structures to two pre-symplectic structures in the space of projective polygons, and study the properties at that level.
Theorem

(Calini, MB, 2017) Given a Hamiltonian $f$ on the moduli space, there exists moduli coordinates $a = (a_n)$ and a vector field $Y^f$ such that if $(u_n)_t = Y_n^f$, then $(a_n)_t$ is $f$-Hamiltonian wrt to the main reduced bracket $\{,\}_1$.

Furthermore, there exists pre-symplectic forms $\omega_1$ and $\omega_2$ on the space of polygons such that

$$\omega_1(Y^f, Y^g)(u) = \{f, g\}_1(a), \quad \omega_2(Y^f, Y^g)(u) = \{f, g\}_2(a).$$
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If $u_n \in \mathbb{RP}^m$ lifts to $V_n \in \mathbb{R}^m$ with $\text{det}(V_n, V_{n+1}, \ldots, V_{n+m-1}) = 1$ for all $n$, 

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$$\omega_1(X, Y)(u) = \frac{1}{2} \sum_{n} \sum_{r=1}^{m-1} \{\det(J_n(Y), V_{n+1}, \ldots, X_{n+r}, \ldots, V_{n+m-1})$$

$$\quad - \det(J_n(X), V_{n+1}, \ldots, Y_{n+r}, \ldots, V_{n+m-1})$$

$$\quad + \det(V_n, \ldots, Y_{n+r-1}, \ldots, X_{n+m-1}) - \det(V_n, \ldots, X_{n+r-1}, \ldots, Y_{n+m-1})\}$$
\[ \omega_2(X, Y) = \frac{1}{2} \sum_{n} \sum_{r=1}^{m-1} \{ \det(Y_n, \ldots, X_{n+r}, \ldots, \gamma_{n+m-1}) \} - \det(X_n, \ldots, Y_{n+r}, \ldots, \gamma_{n+m-1}) \} \].
\[ \omega_2(X, Y) = \frac{1}{2} \sum_n \sum_{r=1}^{m-1} \left\{ \det(Y_n, \ldots, X_{n+r}, \ldots, \gamma_{n+m-1}) - \det(X_n, \ldots, Y_{n+r}, \ldots, \gamma_{n+m-1}) \right\}. \]

Trivially, \( \omega_1(gX, gY) = \omega_1(X, Y) \) and \( \omega_2(gX, gY) = \omega_2(X, Y) \).
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**Theorem**

*(Calini, MB, 17)* \{, \}_1 and \{, \}_2 are compatible.
\[ \omega_2(X, Y) = \frac{1}{2} \sum_n \sum_{r=1}^{m-1} \{ \det(Y_n, \ldots, X_{n+r}, \ldots, \gamma_{n+m-1}) \}
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**Theorem**
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**Theorem**
*(Calini, MB, 18)* There exist two integrable hierarchies associated to each of two vector fields generating the kernel of \( \omega_2 \). Restricted to the moduli space, \( \omega_1 \) is symplectic and one can generate a recursion operator.
GRACIAS!

THANKS!