Noncommutative gauge theories
through twist deformation quantization

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Gel’fand Naimark Theorem (1943)
The notion of a noncommutative space

NCG is based on the correspondence between

\[
\begin{array}{c}
\{ \text{compact Hausdorff spaces} \} \\
\{ \text{commutative unital } C^* \text{algebras} \}
\end{array}
\]^{\text{op}}

\[X \to \mathcal{C}(X)\]

\(\mathcal{C}(X) = \text{algebra of continuous complex valued functions on } X \text{ with pointwise multiplication, involution } f \mapsto f^*, f^*(x) = \overline{f(x)} \text{ and } \|f\|_{\infty} := \sup_{x \in X} |f(x)|.\)

\[\hat{A} = \{ \chi : A \to \mathbb{C} \text{ character} \} \leftarrow A\]
Motivated by Gelfand-Naimark theorem

\[
\left\{ \begin{array}{c}
NC \text{ compact} \\
Hausdorff \text{ spaces}
\end{array} \right\} := \left\{ \begin{array}{c}
\text{(not necessarily commutative)} \\
\text{unital } C^* \text{-algebras}
\end{array} \right\}^{op}
\]

- While a commutative $C^*$-algebra has many characters, one for each point of the underlying space, for a noncommutative $C^*$-algebra characters can be fairly scarce $\implies$ NCG is a "point-free" geometry.

- some spaces are better studied by examining algebras of functions on them;
- in part inspired by quantum mechanics:
  from commutative algebras of classical observables ( = functions on a space )
  to noncommutative algebras of quantum observables ( = operators on a Hilbert space).
More in general, a NC space is an algebra equipped with some additional structures 
\((C^*, \text{von Neumann, quantum group, spectral triple, ...})\)

Example: NC 2-sphere [Podleś]. \(*\)-algebra \(\mathcal{O}(S^2_q)\) generated by elements \(a, a^*, b = b^*\) 
subject to the relations

\[
aa^* + q^{-4}b^2 = 1, \quad a^*a + b^2 = 1; \quad ab = q^{-2}ba, \quad a^*b = q^2ba^*, \quad q \in \mathbb{R}
\]

When \(q = 1\) one recovers the classical commutative algebra \(\mathcal{O}(S^2)\) of polynomials functions on \(S^2\).

NCG has classical geometry (expressed in algebraic terms) as its classical limit.
Bundle theory in noncommutative geometry

Serre-Swan Theorem (1962): the notion of a noncommutative vector bundle

\[
\begin{aligned}
\left\{ \text{vector bundles over } X \right\} & \cong \left\{ \text{projective } \mathcal{C}(X) - \text{modules of finite type} \right\} \\
E & \longrightarrow \mathcal{C}(X)\text{-module } \Gamma(E) \text{ of sections} \\
E_p = \{(x, v) \in X \times \mathbb{C}^N | p(x)v = v\} & \leftarrow \mathcal{E}_p \cong p(\mathcal{C}(X) \otimes \mathbb{C}^N)
\end{aligned}
\]
Bundle theory in noncommutative geometry

Serre-Swan Theorem (1962): the notion of a noncommutative vector bundle

\[ \left\{ \text{vector bundles over } X \right\} \cong \left\{ \text{projective } \mathcal{C}(X) - \text{modules of finite type} \right\} \]

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\[ \left\{ \text{vector bundles over a nc space } A \right\} := \left\{ \text{projective } A - \text{modules of finite type} \right\} \]
Bundle theory in noncommutative geometry
Serre-Swan Theorem (1962): the notion of a noncommutative vector bundle

\[
\begin{align*}
\left\{ \text{vector bundles over } X \right\} & \cong \left\{ \text{projective } \mathcal{C}(X) - \text{modules of finite type} \right\} \\
E & \longrightarrow \mathcal{C}(X)\text{-module } \Gamma(E) \text{ of sections} \\
E_p = \{(x, \nu) \in X \times \mathbb{C}^N | p(x)\nu = \nu\} & \longleftarrow \mathcal{E}_p \cong p(\mathcal{C}(X) \otimes \mathbb{C}^N)
\end{align*}
\]

\[
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\left\{ \text{vector bundles over a nc space } A \right\} & := \left\{ \text{projective } A - \text{modules of finite type} \right\}
\end{align*}
\]

Remark: Finite projective modules correspond to idempotents in matrix algebras:

\[
\begin{align*}
\mathcal{E} \text{ finite projective over } A & \iff \exists N \in \mathbb{N} / A \otimes \mathbb{C}^N = \mathcal{E} \oplus \mathcal{E}' \\
& \iff \exists N \in \mathbb{N}, p = p^2 = p^* \in \text{Mat}_N(A) / \mathcal{E} \cong p(A \otimes \mathbb{C}^N)
\end{align*}
\]
Example. Podleś 2-sphere $S^2_q$ with generators $a, a^*, b = b^*$ subject to the relations

$$aa^* + q^{-4}b^2 = 1, \quad a^* a + b^2 = 1; \quad ab = q^{-2}ba, \quad a^* b = q^2 ba^*, \quad q \in \mathbb{R}^+$$

The matrix

$$p_q := \frac{1}{2} \begin{pmatrix} 1 + q^{-2}b & q a \\ q^{-1}a^* & 1 - b \end{pmatrix} \in \text{Mat}(2, S^2_q)$$

is an idempotent $\twoheadrightarrow$ vector bundle over $S^2_q$ (monopole)

($p_q$ is the Bott projection in the classical limit $q = 1$).
Example. Podleś 2-sphere $S^2_q$ with generators $a, a^*, b = b^*$ subject to the relations

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The matrix

$$pq := \frac{1}{2} \begin{pmatrix} 1 + q^{-2}b & qa \\ qa^* & 1 - b \end{pmatrix} \in \text{Mat}(2, S^2_q)$$

is an idempotent $\leadsto$ vector bundle over $S^2_q$ (monopole) ($pq$ is the Bott projection in the classical limit $q = 1$).

- notion of equivalence of idempotents $\leadsto$ K-theory;
- topological invariants (via Connes-Chern pairing);
- purely algebraic def. of differential calculus $(\Omega^n A, d)$ on a (nc) algebra $A$;
- connection on $\mathcal{E}$ is $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1 A$ satisfying Leibniz rule (for $\mathcal{E}$ finite projective module, always $\exists$ Grassmann connection: $\nabla := pd$);
- $\nabla^2$ curvature $\leadsto$ gauge theories on NC spaces.
Symmetries: from groups to Hopf algebras.

$G$ group of matrices ($G = SL(n, \mathbb{C}),\ SO(n, \mathbb{C}), \ldots$)

\[
\begin{align*}
\mu : G \times G &\to G, \ (g, h) \mapsto gh \\
e &\in G \\
\text{inv} : G &\to G, \ g \mapsto g^{-1}
\end{align*}
\]

$\mapsto \mathcal{O}(G)$ is a Hopf algebra: unital algebra $H$ with

\[
\Delta : H \to H \otimes H \text{ coproduct } , \quad h \mapsto h_{(1)} \otimes h_{(2)}
\]

$\varepsilon : H \to \mathbb{C} \text{ counit}$

$S : H \to H \text{ antipode}$

satisfying prop. 1 – 3 below.

indeed the group structure induces on $H := \mathcal{O}(G)$ the maps

\[
\Delta = \mu^* \quad \varepsilon = \text{ev}_e \quad S = \text{inv}^*
\]

1. $\mu$ associative $\Rightarrow (\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

2. $ge = g = eg$ $\Rightarrow (\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$

3. $gg^{-1} = e = g^{-1}g$ $\Rightarrow m(S \otimes id)\Delta = \varepsilon(1) = m(id \otimes S)\Delta$
The theory of Hopf algebras has its roots in algebraic topology (Hopf ’40s, Sweedler ’60s). Later in the 80’s: quantum group theory (Faddeev-Reshetikhin-Takhtajan, Drinfeld, Woronowicz, Majid,...).

- Coordinate algebras of quantum groups (FRT bialgebras): $SO_q(n)$, $U_q(n)$, $Sp_q(2n)$, ..., $SU_q(2)$ (Woronowicz)
- Quantized universal enveloping algebras (Drinfeld-Jimbo algebras): $U_q(g)$
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- Coordinate algebras of quantum groups (FRT bialgebras): \(SO_q(n), \ U_q(n), \ Sp_q(2n),\ldots\); \(SU_q(2)\) (Woronowicz)
- Quantized universal enveloping algebras (Drinfeld-Jimbo algebras): \(\mathcal{U}_q(\mathfrak{g})\)

---

**Definition**

Let \(H\) be a Hopf algebra, an \(H\)-comodule algebra is an algebra \(A\) together with an algebra morphism \(\delta : A \to A \otimes H\) (coaction) such that

\[(\Delta \otimes \text{id})\delta = (\text{id} \otimes \delta)\delta\ , \ (\varepsilon \otimes \text{id})\delta = \text{id}\]

**\(G\)-spaces:** \(\alpha : X \times G \to X\) action dualizes to \(\delta := \alpha^* : \mathcal{C}(X) \to \mathcal{C}(X) \otimes \mathcal{C}(G)\)

\[x(gh) = (x(g))h \quad \sim \quad (\text{id} \otimes \Delta) \circ \delta = (\delta \otimes \text{id}) \circ \delta\]

\[(x)e = x \quad (\text{id} \otimes \varepsilon)\delta = \text{id}\]
Noncommutative gauge theories through twist deformation quantization

NC principal bundles & Hopf-Galois extensions [Kreimer, Takeuchi 1981]

- $H$ Hopf algebra (structure group)
- $A$ an $H$-comodule algebra (total space) with coaction the algebra map
  \[ \delta : A \rightarrow A \otimes H, \ a \mapsto a_{(0)} \otimes a_{(1)} \]
- $B$ algebra (base space), $B \simeq A^{\text{co}(H)} := \{ b \in A | \delta(b) = b \otimes 1_H \}$
- Principality condition: the algebra extension $B \subseteq A$ is Hopf-Galois:
  \[ \chi = (m_A \otimes id)(id \otimes_B \delta) : A \otimes_B A \rightarrow A \otimes H \]
  \[ a \otimes_B a' \mapsto a a'_{(0)} \otimes a'_{(1)} \]
  (canonical map) is bijective.
Example: the 2\textsuperscript{nd} Hopf bundle (instanton bundle)

\[ S^7 \times SU(2) \xrightarrow{\alpha} S^7 \]

\[ S^4 \cong S^7 / SU(2) \]

\[ A = O(S^7) \xleftarrow{\delta = \alpha^*} O(S^7) \otimes O(SU(2)) \]

\[ B = O(S^4) \cong O(S^7)^{co H} \]

principal bundle

Hopf-Galois extension
Example: the 2\textsuperscript{nd} Hopf bundle (instanton bundle)

\[ S^7 \times SU(2) \xrightarrow{\alpha} S^7 \]
\[ S^4 \cong S^7 / SU(2) \]

\[ A = O(S_q^7) \xleftarrow{\delta} O(S_q^7) \otimes O(SU_q(2)) \]

\[ B = O(S_q^4) \cong O(S_q^7)^{co H} \]

principal bundle \hspace{5cm} (family of) Hopf-Galois extensions
Example: the 2\textsuperscript{nd} Hopf bundle (instanton bundle)

\[
S^7 \times SU(2) \xrightarrow{\alpha} S^7 \quad \quad \quad \quad A = \mathcal{O}(S^7_q) \xleftarrow{\delta} \mathcal{O}(S^7_q) \otimes \mathcal{O}(SU_q(2))
\]

\[
S^4 \cong S^7 / SU(2) \quad \quad \quad \quad B = \mathcal{O}(S^4_q) \cong \mathcal{O}(S^7_q)^{co H}
\]

principal bundle \quad \quad \quad \quad (family of) Hopf-Galois extensions

- various constructions on different noncommutative spheres, e.g.
  - from FRT-bialgebras with $S^7_q$ as quantum homog. space of $\mathcal{O}(Sp_q(2))$. Here $q \in \mathbb{R}$, with $q = 1$ classical case. [Landi, P., Reina 2006]
  - or isospectral deformations $\mathcal{O}(S^m_{\Theta})$. Here $\Theta \in \text{Mat}(n, \mathbb{R})$, $m = 2n, 2n + 1$ antisymmetric, with $\Theta = 0$ classical case. [Landi, Brain, P., Reina, van Suijlekom, ... 2005–]
- monopole bundle $S^3 \rightarrow S^2$ [Brzezinski, Majid, 1993]
- associated vector bundles $p = p^2$ and (instanton) connections $\nabla = pd$. 
use the theory of Drinfeld to deform algebra extensions into new algebra extensions in such a way to preserve the condition to be Hopf-Galois, i.e. the invertibility of the canonical map

$$\chi : A \otimes_B A \rightarrow A \otimes H \in \text{Mor}(A, \mathcal{M}_A^H)$$

deform (classical or nc) principal bundles into (nc) principal bundles.
Drinfel’d theory of twists

Definition

A linear map $\gamma : H \otimes H \rightarrow \mathbb{K}$ is called a **(unital) 2-cocycle** on $H$ provided

$$
\gamma \left( g(1) \otimes h(1) \right) \gamma \left( g(2) h(2) \otimes k \right) = \gamma \left( h(1) \otimes k(1) \right) \gamma \left( g \otimes h(2) k(2) \right)
$$

$$
\gamma \left( h \otimes 1_H \right) = \varepsilon(h) = \gamma \left( 1_H \otimes h \right)
$$

for all $g, h, k \in H$ (where $h(1) \otimes h(2) = \Delta(h)$ coproduct, sum understood).

Twisting Hopf-algebras:

Let $\gamma$ be a convolution invertible 2-cocycle on $(H, \Delta, \varepsilon)$ with inverse $\tilde{\gamma}$. Then

$$
m_\gamma(h \otimes k) := h \cdot_\gamma k := \gamma \left( h(1) \otimes k(1) \right) h(2) k(2) \tilde{\gamma} \left( h(3) \otimes k(3) \right)
$$

defines a new associative product on (the $\mathbb{K}$-module underlying) $H$.

The resulting algebra $H_\gamma := (H, m_\gamma, 1_H)$ with unchanged coproduct $\Delta$ and counit $\varepsilon$ and twisted antipode $S_\gamma := u_\gamma \ast S \ast \tilde{u}_\gamma$ is a Hopf algebra.
Deforming spaces carrying $H$ as a symmetry:

\[(A, \delta^A) \in A^H \quad \rightsquigarrow \quad (A_\gamma, \delta^{A_\gamma}) \in A^{H_\gamma}\]

If \((A, \delta^A) \in A^H\) is a right $H$-comodule algebra with coaction

\[\delta^A : A \to A \otimes H, \quad a \mapsto a_{(0)} \otimes a_{(1)}\]

then $A$, with same coaction, is an $H_\gamma$-comodule algebra when endowed with the new product

\[a \otimes a' \mapsto a \blacktriangledown a' := a_{(0)} a'_{(0)} \tilde{\gamma} (a_{(1)} \otimes a'_{(1)})\]

We denote it by $A_\gamma$. 
Twisting of Hopf-Galois extensions

**Case 1: cocycle** $\gamma : H \otimes H \to \mathbb{K}$ on the ‘structure group’ $H$

- $H \rightsquigarrow$ twisted Hopf-algebra $H_\gamma$
  
  with twisted product $h \cdot_{\gamma} k := \gamma(h(1) \otimes k(1)) h(2) k(2) \tilde{\gamma}(h(3) \otimes k(3))$

- $A \in \mathcal{A}^H \rightsquigarrow$ twisted comodule-algebra $A_\gamma \in \mathcal{A}^{H_\gamma}$ with same coaction and twisted product $a \cdot_{\gamma} a' := a(0) a'(0) \tilde{\gamma}(a(1) \otimes a'(1))$

- $B \subseteq A$ is unchanged!

- $\rightsquigarrow$ apply to HG extensions:

\[
\begin{array}{c}
A \\
H \uparrow \\
B = A^{coH}
\end{array}
\quad \rightsquigarrow \gamma \text{ on } H \quad \rightsquigarrow
\begin{array}{c}
A_\gamma \\
H_\gamma \uparrow \\
B = A_\gamma^{coH_\gamma}
\end{array}
\]
Theorem

The following diagram in $A^\gamma \mathcal{M}^H_{A^\gamma}$ commutes:

\[
\begin{array}{ccc}
A^\gamma \otimes_B^\gamma A^\gamma & \xrightarrow{\chi^\gamma} & A^\gamma \otimes^\gamma (H^\gamma) \\
\downarrow \varphi_{A,A} & & \downarrow \text{id} \otimes^\gamma Q \\
(A \otimes_B^\gamma A)^\gamma & \xrightarrow{\Gamma(\chi) = \chi} & (A \otimes^\gamma H)^\gamma \\
\end{array}
\]

Corollary

The extension $B = A^{\text{co}H} \subset A$ is $H$-Galois $\iff$ the extension $B \simeq A^{\text{co}H^\gamma}_{A^\gamma} \subset A^\gamma$ is $H^\gamma$-Galois.
Case 2: cocycle $\sigma$ on an external Hopf algebra of symmetries

Let $K$ be a Hopf algebra and $\sigma$ a 2-cocycle on it.

Suppose that the total space $A$ carries an additional structure of left $K$-comodule algebra $A \in {}^K A$ s.t. the coaction $\rho^A : A \to K \otimes A$ is $H$-equivariant:

$$(\rho^A \otimes \text{id})\delta^A = (\text{id} \otimes \delta^A)\rho^A$$

- $\sigma A$ still carries the coaction of $H$!
- the base space $B$ is twisted! (while $H$ is unchanged)

**Theorem**

$B \subseteq A$ is Hopf-Galois if and only if $\sigma B \subseteq \sigma A$ is Hopf-Galois.
EXAMPLE. The quantum Hopf bundle on the Connes-Landi sphere $S^4_{\theta}$

- Let $K = O(\mathbb{T}^2)$ be the (commutative) algebra of functions on the 2-torus $\mathbb{T}^2$, $\exists$ a left coaction of $O(\mathbb{T}^2)$ on the algebra $O(S^7)$:

$$\rho : O(S^7) \to O(\mathbb{T}^2) \otimes O(S^7), \quad z_i \mapsto \tau_i \otimes z_i$$

which is $O(SU(2))$-equivariant.

- Let $\sigma$ be the exponential 2-cocycle on $O(\mathbb{T}^2)$ determined by setting

$$\sigma\left(t_j \otimes t_k\right) = \exp(i\pi \Theta_{jk}); \quad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}; \quad \theta \in \mathbb{R}$$

The resulting bundle is the quantum Hopf bundle on the Connes-Landi sphere $O(S^4_{\theta})$ [Landi, van Suijlekom, 2005].

**Remark:** Its principality follows from the theory and doesn’t need to be proved!
Case 3: combination of deformations

Case 1. $\gamma$ on $H$: $(A, H, B) \mapsto (A_\gamma, H_\gamma, B)$  
Case 2. $\sigma$ on $K$: $(A, H, B) \mapsto (\sigma A, H, \sigma B)$

- Let as before $A$ be a right $H$-comodule algebra with an equivariant left coaction of $K$
- Let $\gamma$ a 2-cocycle on $H$ and $\sigma$ a 2-cocycle on $K$

\[
\begin{array}{c}
A \\
\downarrow H \\
B
\end{array} \quad \xrightarrow{\text{double twisting}} \quad \begin{array}{c}
\sigma A_\gamma \\
\downarrow H_\gamma \\
\sigma B
\end{array}
\]

**Theorem**

$B \subseteq A$ is $H$-Hopf Galois if and only if $\sigma B \subseteq \sigma A_\gamma$ is $H_\gamma$-Hopf Galois.

**Application:** quantum homogeneous spaces
The gauge group

For a principal $G$-bundle $\pi : P \to X$, the group $\mathcal{G}_P$ of gauge transformations is

- the subgroup of principal bundle automorphisms which are vertical:
  
  $\mathcal{G}_P = \text{Aut}_V(P) := \{ \varphi : P \to P; \varphi(pg) = \varphi(p)g, \pi \circ \varphi = \pi \}$

  with group law given by the composition of maps;

- the group of $G$-equivariant maps,

  $\mathcal{G}_P = \{ \sigma : P \to G; \sigma(pg) = g^{-1}\sigma(p)g \}$

  with pointwise product, $(\sigma \cdot \tau)(p) := \sigma(p)\tau(p) \in G$.

  (Locally, $x \in X \to g(x) \in G$)
The group of gauge transformations acts by pullback on the set $A_P$ of connections of the bundle $\pi : P \to X$.

$\omega, \eta$ connection forms are gauge equivalent iff $\exists \varphi \in G_P$ such that $\varphi^* \omega = \eta$.

Indeed gauge equivalence defines an equivalence relation on $A_P$

$$\sim \Rightarrow \mathcal{M} = A_P/G_P \quad \text{moduli space of connections}$$
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**Aim:** extend the notion of gauge transformations to the algebraic framework of (NC) Hopf-Galois extensions.

- [Brzeziński (1996)]

**Problem:** In the classical limit (commutative case) it doesn’t give the expected result, but a group bigger than the gauge group of the bundle....

- [Aschieri, Landi, P. (2018)] in the framework of coquasitriangular Hopf algebras
Coquasitriangular Hopf algebras

A Hopf algebra $H$ is called coquasitriangular if it is endowed with a linear map

$$R : H \otimes H \to \mathbb{K} \quad \text{(universal r-form)}$$

(with some properties) such that $m_{op} = R \ast m \ast \bar{R}$, i.e. for all $h, k \in H$

$$kh = R\left(h_{(1)} \otimes k_{(1)}\right)h_{(2)}k_{(2)}\bar{R}\left(h_{(3)} \otimes k_{(3)}\right)$$

Examples

- commutative Hopf algebras with trivial universal $r$-form $R = \varepsilon \otimes \varepsilon$;
- the noncommutative FRT bialgebras $O_q(G)$ deformations of the algebras of coordinate functions on Lie groups;
- 2-cocycle deformations of coquasitriangular Hopf algebra $(H, R)$ with universal $r$-form

$$R_\gamma := \gamma_{21} \ast R \ast \bar{\gamma} : h \otimes k \mapsto \gamma\left(k_{(1)} \otimes h_{(1)}\right)R\left(h_{(2)} \otimes k_{(2)}\right)\bar{\gamma}\left(h_{(3)} \otimes k_{(3)}\right)$$
Some useful facts from the theory of cqt Hopf algebras:

- The category $\mathcal{A}_H^H, \boxtimes$ of $H$-comodule algebras is monoidal: $(A, \delta^A), (C, \delta^C) \in \mathcal{A}_H^H$, then the $H$-comodule $A \otimes C$ with tensor product coaction
  \[ \delta^{A\otimes C} : a \otimes c \mapsto a_{(0)} \otimes c_{(0)} \otimes a_{(1)} c_{(1)} \]
is a right $H$-comodule algebra,

\[ A \boxtimes C := (A \otimes C, \bullet) \quad \text{(braided product algebra)} \]

when endowed with the product

\[ (a \otimes c) \bullet (a' \otimes c') := a \, R_{C,A}(c \otimes a') c' = a \, a'_{(0)} \otimes c_{(0)} c' \, R\left(c_{(1)} \otimes a'_{(1)}\right) . \]

- The right $H$-comodule $\underline{H} = (H, \text{Ad})$ becomes an $H$-comodule algebra $\underline{H} = (H, \star, \text{Ad})$ when endowed with the product

\[ h \star k := h_{(2)} k_{(2)} R\left(S(h_{(1)}) h_{(3)} \otimes S(k_{(1)})\right) \]

$(\underline{H}, \star, \eta, \Delta, \epsilon, S, \text{Ad})$ is a braided Hopf algebra (associated with $H$).
Hopf-Galois extensions for coquasitriangular Hopf algebras and their gauge groups. [P. Aschieri, G. Landi, C.P. (2018)]

Theorem

Let \((H, R)\) be a coquasitriangular Hopf algebra and \(A \in A_{qc}^{(H,R)}\) a quasi-commutative \(H\)-comodule algebra. Let \(B \subseteq Z(A)\) be the corresponding subalgebra of coinvariants. Then the canonical map

\[
\chi = (m \otimes \text{id}) \circ (\text{id} \otimes_B \delta^A) : A \otimes_B A \longrightarrow A \otimes H, \\
\quad a' \otimes_B a \longmapsto a' a_{(0)} \otimes a_{(1)}
\]

is an algebra map, thus a morphism in \(A^H\).

Definition

Let \((H, R)\) be a coquasitriangular Hopf algebra. A right \(H\)-comodule algebra \(A\) is quasi-commutative (with respect to the universal \(r\)-form \(R\)), \(A \in A_{qc}^{(H, R)}\) if

\[
m_A = m_A \circ R_{A,A}, \quad ac = c_{(0)} a_{(0)} R(a_{(1)} \otimes c_{(1)}) \quad a, c \in A
\]
Examples

- Clearly, for \((H, \varepsilon \otimes \varepsilon)\), every commutative algebra \(A \in A^H\) is quasi-commutative.
- Twist deformations \(A_{\gamma} \in A^{H_{\gamma}}\) of quasi-commutative algebras \(A\) via a 2-cocycle on \(H\) are quasi-commutative algebras.
- A main example of quasi-commutative comodule algebra is the \(H\)-comodule algebra \((H, \star, \text{Ad})\) associated with a cotriangular Hopf algebra \((H, R)\).
- \(H = \mathcal{O}(GL_q(2))\) is coquasitriangular with (not cotriangular) universal \(r\)-form

\[
R(u_{ij} \otimes u_{kl}) = q^{-1}R^{ik}_{jl}, \quad R(D^{-1} \otimes u_{ij}) = R(u_{ij} \otimes D^{-1}) = q \delta_{ij},
\]

The quantum plane \(\mathcal{O}(\mathbb{C}_q^2) = \mathbb{C}[x, x_2]/\langle x_1 x_2 - q x_2 x_1 \rangle\) is a quasi-commutative \(\mathcal{O}(GL_q(2))-\)comodule algebra with coaction \(\delta(x_i) = \sum_j x_j \otimes u_{ji}\).
The gauge group of a (coquasi\(\triangle\)) Hopf-Galois extension.

Let \(B \subseteq A \in \mathcal{A}_{qc}^{(H,R)}\) be an \(H\)-Hopf-Galois extension, with \(H\) coquasitriangular.

**Theorem**

The \(K\)-module of left \(B\)-module, right \(H\)-comodule algebra morphisms

\[
\text{Aut}_V(A) := \text{Hom}_{BAH}(A,A) = \{ \mathcal{F} \in \text{Hom}_{AH}(A,A), \text{such that } \mathcal{F}|_B = \text{id}\}
\]

is a group with respect to map composition \(\mathcal{F} \cdot G := G \circ \mathcal{F}\).
The gauge group of a (coquasi\(\Delta\)) Hopf-Galois extension.

Let \(B \subseteq A \in \mathcal{A}_{qc}^{(H,R)}\) be an \(H\)-Hopf-Galois extension, with \(H\) coquasitriangular.

**Theorem**

*The \(\mathbb{K}\)-module of left \(B\)-module, right \(H\)-comodule algebra morphisms*

\[
\text{Aut}_V(A) := \text{Hom}_{B,\mathcal{A}_H}(A, A) = \{F \in \text{Hom}_{\mathcal{A}_H}(A, A), \text{such that } F|_B = \text{id}\}
\]

is a group with respect to map composition \(F \cdot G := G \circ F\).

**Theorem**

*The \(\mathbb{K}\)-module of \(H\)-equivariant algebra maps \(H \rightarrow A\)*

\[
\mathcal{G}_A := \text{Hom}_{\mathcal{A}_H}(H, A)
\]

is a group with respect to the convolution product, with inverse \(\overline{f} := f \circ S\) for \(f \in \text{Hom}_{\mathcal{A}_H}(H, A)\). Moreover, the groups \((\mathcal{G}_A, \ast)\) and \((\text{Aut}_V(A), \cdot)\) are isomorphic.
Twisting gauge groups

**Theorem**

Let $B = A^{coH} \subseteq A$ be a Hopf-Galois extension and $\gamma$ a 2-cocycle on $H$, with $H$ coquasitriangular and $A \in A_{qc}^{(H,R)}$. The gauge group $G_{A,\gamma}$ of the twisted Hopf-Galois extension $B = A^{coH,\gamma} \subseteq A_\gamma \in A_{qc}^{(H,\gamma,R,\gamma)}$ is isomorphic to the gauge group $G_A$ of the initial Hopf-Galois extension.
Twisting gauge groups

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Next

- gauge group of a Hopf-Galois extension obtained by twisting for a 2-cocycle on an external Hopf algebra of symmetries $K$ (e.g. instanton bundle on Connes-Landi sphere $S^4_\theta$)

- Gauge group of a generic Hopf-Galois extension? Group or Hopf algebra?