An introduction to global class field theory
(Towards a $p$-adic Langlands correspondence)

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Notation

Let $k$ be a number field, $\mathcal{O}$ its ring of integers.

A place is an equivalence class of absolute values, called finite (whenever they are non–archimedean) or infinite (otherwise) Let $P_k$ be the set of places of $k$. 
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Let $p \in P_k$ (either finite or infinite). We’ve got:

a) $k_p$, the completion (which must be $\mathbb{R}$, $\mathbb{C}$ or a $p$–adic one).

b) $\mathcal{O}_p = \{\alpha \in k_p^* \mid |\alpha|_p \leq 1\}$, the ring of integers of $k_p$.

c) $U_p = \{\alpha \in k_p^* \mid |\alpha|_p = 1\}$, the group of units
The ring of adèles of $k$, noted $\mathbb{A}_k$ is

$$\mathbb{A}_k = \left\{ (\alpha_p)_{p \in P_k} \mid \alpha_p \in \mathcal{O}_p \text{ for almost all } p \in P_k \right\}$$
The ring of adèles of $k$, noted $A_k$ is

$$A_k = \left\{ (\alpha_p)_{p \in P_k} \mid \alpha_p \in \mathcal{O}_p \text{ for almost all } p \in P_k \right\}$$

This is also called the *restricted product* of the $k_p$ w.r.t $\mathcal{O}_p \subset k_p$.

It is a ring (adding and multiplying componentwise).
Adèles

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Easy example:

$$\mathbb{A}_Q = \mathbb{R} \times \left\{ (a_p) \mid a_p \in \mathbb{Q}_p \text{ and } a_p \in \mathbb{Z}_p \text{ for almost all } p \right\}.$$
Idèles (I)

The idèles of $k$, noted $\mathbf{I}_k$, is the unit group of $\mathbf{A}_k$ (which would usually be noted $\mathbf{A}_k^*$).

They may also be described as the restricted product of $k_p^*$ w.r.t. $U_p$. 
The idèles of $k$, noted $I_k$, is the unit group of $A_k$ (which would usually be noted $A^*_k$).

They may also be described as the restricted product of $k^*_p$ w.r.t. $U_p$.

$k \hookrightarrow k_p$ induces a diagonal embedding

$$K^* \hookrightarrow I_k,$$

associating $a \in k^*$ with $(\alpha_p)$ which is $a$ at each $p$-component
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$k \hookrightarrow k_p$ induces a diagonal embedding

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associating $a \in k^*$ with $(\alpha_p)$ which is $a$ at each $p$-component

Such elements are called principal idèles, they are a subgroup of $\mathbf{I}_k$ and the quotient

$$C_k = \mathbf{I}_k / k^*$$

is called the idèle class group. Its elements will be noted $[\alpha]$. 
If $S \subset P_k$ is a finite set of places, we call

$$I_k^S = \prod_{p \in S} k_p^* \times \prod_{p \notin S} U_p$$

the group of $S$–idèles, which is obviously a subgroup of $I_k$. 
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For instance, if $S_\infty$ is the set of infinite places

$$I_k^{S_\infty} = \prod_{p \mid \infty} k_p^* \times \prod_{p \text{ finite}} U_p,$$

where the first factors are either $\mathbb{R}^*$ or $\mathbb{C}^*$. 
Let \( \text{FId}_k \) be the group of fractional ideals of \( k \).
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We can define

$$I_k \rightarrow \text{FId}_k$$

$$(\alpha_p) \rightarrow \prod_{p \text{ finite}} p^{v_p(\alpha_p)}$$
Let $\text{Fld}_k$ be the group of fractional ideals of $k$.

We can define

\[
I_k \longrightarrow \text{Fld}_k \\
(\alpha_p) \longmapsto \prod_{p \text{ finite}} p^{v_p(\alpha_p)}
\]

It is a surjective homomorphism, with kernel $I_k^{S_\infty}$.
Therefore we have an isomorphism

\[ \mathcal{I}_k / \mathcal{I}_k^{S_\infty} \cong \text{FId}_k. \]
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In this isomorphism principal idèles correspond to principal fractional ideals, and vice versa. Hence

\[ \mathcal{I}_k / \left( k^* \cdot \mathcal{I}_k^{S_\infty} \right) \cong \text{Cl}_k \]
Norm (I)

Let $\alpha = (\alpha_p) \in \mathfrak{I}_k$. We define

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Let \( \alpha = (\alpha_p) \in I_k \). We define

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| (\alpha_p) |_p = | \alpha_p |_p,
\]

and, subsequently,

\[
| \alpha | = \prod_{p} | \alpha_p |_p.
\]

We can assume

- \( p \) real \( \rightarrow \) \( | \cdot | \)
- \( p \) complex \( \rightarrow \) \( | \cdot |^2 \)
- \( p \) is over \( p \) \( \rightarrow \) \( |p|_p = 1/p \)
Let $\alpha = (\alpha_p) \in I_k$. We define

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$$|\alpha| = \prod_p |\alpha_p|_p.$$ 

We can assume

- $p$ real $\rightarrow$ $|\cdot|$ 
- $p$ complex $\rightarrow$ $|\cdot|^2$ 
- $p$ is over $p$ $\rightarrow$ $|p|_p = 1/p$

And, because of the product formula,

$$x \in k^* \rightarrow |x| = 1.$$
Norm (II)

We have defined then a *norm* mapping

\[ | \cdot | : I_k \rightarrow \mathbb{R}^* \]

which is a surjective group homomorphism (define an “inverse”).
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\[ | \cdot | : \mathbb{I}_k \longrightarrow \mathbb{R}^*_+ \]

which is a surjective group homomorphism (define an “inverse”).

We call its kernel

\[ \mathbb{I}^0_k = \{ \alpha \in \mathbb{I}_k \mid |\alpha| = 1 \} \]

which verifies \( k^* \subset \mathbb{I}^0_k \).
We have defined then a norm mapping

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which is a surjective group homomorphism (define an “inverse”).

We call its kernel

\[ I_k^0 = \{ \alpha \in I_k \mid |\alpha| = 1 \} \]

which verifies \( k^* \subset I_k^0 \).

Therefore we can consider a norm (induced, and identically noted) on the idèle class group:

\[ | \cdot | : C_k \rightarrow \mathbb{R}_+^* \]

whose kernel, noted \( C_k^0 \) will be of some interest.
Topological groups

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Big advantage: 1 is (almost) all that matters for local (and sometimes global) issues.
Not really much choice

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More specifically, we would like $I^S_k$ to be open subgroups whenever $S_\infty \subset S$. But then
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More specifically, we would like $I_k^S$ to be open subgroups whenever $S_\infty \subset S$. But then

**Theorem.**– There exists a unique topology in $I_k$ such that, if $S_\infty \subset S$ and $S$ is finite, $I_k^S$ is open.
Definition via neighbourhoods of 1

This topology, when defined by basic systems of neighbourhoods, is given (at 1) by

$$\prod_{p \in S} W_p \times \prod_{p \notin S} U_p$$
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where \( W_p \) is a basic system of neighbourhoods of \( 1 \in k_p \), and \( S \) is finite, \( S_\infty \subset S \).
Definition via neighbourhoods of 1

This topology, when defined by basic systems of neighbourhoods, is given (at 1) by

$$\prod_{p \in S} W_p \times \prod_{p \not\in S} U_p$$

where $W_p$ is a basic system of neighbourhoods of $1 \in k_p$, and $S$ is finite, $S_\infty \subset S$.

Equivalently we can take

$$N(S, \epsilon) = \{(\alpha_p) \mid |\alpha_p|_p = 1 \text{ if } p \not\in S, \quad |\alpha_p - 1|_p < \epsilon \text{ if } p \in S\}.$$
$S$–idèles are closed

Let $p$ be a place. Then the projection

$$I_k \xrightarrow{\pi} k_p$$

is continuous (it is in $I_k^{S\infty}$, therefore in $1$, therefore in $I_k$).
S–idèles are closed

Let $p$ be a place. Then the projection

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Then $\pi^{-1}(U_p)$ is closed.
$S$–idèles are closed

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is continuous (it is in $\mathfrak{I}_k^{\infty}$, therefore in 1, therefore in $\mathfrak{I}_k$).

Then $\pi^{-1}(U_p)$ is closed.

And hence so it is
\[ I_k^S = \bigcap_{p \notin S} \pi^{-1}(U_p). \]
Locally compactness

Let $S_\infty \subset S$ and consider

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The first factor is a finite product of locally compact spaces, while the second is a product of compact spaces.
Locally compactness

Let $S_\infty \subset S$ and consider

$$I^S_k = \prod_{p \in S} k_p^* \times \prod_{p \notin S} U_p.$$ 

The first factor is a finite product of locally compact spaces, while the second is a product of compact spaces.

Hence $I^S_k$ is locally compact, and so it is $I_k$. 
So far, so good

$\mathbf{I}_k$ is a locally compact topological group.
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$I_k$ is a locally compact topological group.

If $S$ is finite, $I^S_k$ is a closed subgroup.
So far, so good

$I_k$ is a locally compact topological group.

If $S$ is finite, $I_k^S$ is a closed subgroup.

If $S_\infty \subset S$, $I_k^S$ is an open subgroup.
$I_k^{S_\infty}$ is open and closed, therefore $\{1\}$ is open and closed in the quotient space $I_k/I_k^{S_\infty}$. 
\( I^S_\infty \) is open and closed, therefore \( \{1\} \) is open and closed in the quotient space \( I_k/I^S_\infty \).

Then \( I_k/I^S_\infty \) must be a discrete space.
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Then $I_k/I_k^{S_{\infty}}$ must be a discrete space.

So, if we consider the discrete topology on $\text{Fld}_k$, we have a homeomorphism

$$I_k/I_k^{S_{\infty}} \cong \text{Fld}_k.$$
A bit more of norm

Remember that our norm homomorphism

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But it is also continuous. It is in \( I_k^{S_{\infty}} \), therefore in \( 1 \), therefore in \( I_k \).
A bit more of norm

Remember that our norm homomorphism

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was surjective.

But it is also continuous. It is in \( I_k^{\infty} \), therefore in 1, therefore in \( I_k \).

Its “inverse” is continuous as well, henceforth we have a homeomorphism

\[ I_k / I_k^0 \cong \mathbb{R}^*_+ \].
A bit more of $k^*$

Remember that $k^*$ could be viewed as a subgroup of $I_k$.

**Proposition.** $k^*$ is a discrete closed subgroup of $I_k$. 
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The proof basically consists of showing that $N(S_\infty, \epsilon) \cap k^* = \{1\}$, hence $k^*$ is discrete.
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The proof basically consists of showing that $N(S_\infty, \epsilon) \cap k^* = \{1\}$, hence $k^*$ is discrete.

As a corollary, we have a locally compact topology in $C_k$.
A bit more of $C_k^0$

$$C_k^0 = \{ [\alpha] \in C_k \mid ||\alpha|| = 1 \}$$

**Proposition.** $C_k^0$ is compact.
A bit more of $C_k^0$

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**Proposition.** $C_k^0$ is compact.

The proof chooses a big enough $\rho > 0$ such that any idèle of such norm is $k^*$-congruent to another whose components are all of smaller norm (yes, you can do that).
A bit more of $C_k^0$

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**Proposition.** $C_k^0$ is compact.

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The set of such idèles is compact, so the original set of idèles with norm $\rho$ is also compact (closed subset) and it is homeomorphic to $C_k^0$. 
Why is important that $C_k^0$ is compact? (I)

Remember we had

$$\mathbb{I}_k / \left( k^* \cdot \mathbb{I}^S_k \right) \simeq \text{Cl}_k$$
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Then, for an element in $\text{Cl}_k$ we have a class $[\alpha] \in C_k$ modulo the projection of $I_k^{S_\infty}$. 
Why is important that $C_k^0$ is compact? (I)

Remember we had

$$I_k / \left( k^* \cdot I_k^{S_\infty} \right) \simeq Cl_k$$

Then, for an element in $Cl_k$ we have a class $[\alpha] \in C_k$ modulo the projection of $I_k^{S_\infty}$.

This means we can pick an idèle $\alpha$ on it in such a way that $|\alpha| = 1$ (adjusting the norm at the infinite places). We have then a map

$$C_k^0 \longrightarrow Cl_k$$

which is surjective.
Why is important that $C^0_k$ is compact? (II)

As $C^0_k$ is compact, so is $\text{Cl}_k$ (for the discrete topology), hence it must be finite.
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As $C_k^0$ is compact, so is $\text{Cl}_k$ (for the discrete topology), hence it must be finite.

Another corollary is:

**Theorem (Dirichlet).**– The group $U_k$ has rank $r + s - 1$ (where $r$ is the number of real places and $s$ is the number of complex places).
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**Theorem (Dirichlet).** The group $U_k$ has rank $r + s - 1$ (where $r$ is the number of real places and $s$ is the number of complex places).

The proof is somehow more involved (lattices and so on).
A word on open subgroups (I)

We will call a finite formal sum

\[ m = \sum_{p \in P_k} n_p p, \]

where \( n_p = 0 \) almost always
\( n_p = 0, 1 \) if \( p \) is real a divisor.
\( n_p = 0 \) if \( p \) is complex
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\[ n_p = 0 \quad \text{if } p \text{ is complex} \]

Let us write \( \text{supp}(m) = \{ p \mid n_p \neq 0 \} \).
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Let us write \( \text{supp}(m) = \{ p \mid n_p \neq 0 \} \).

Variants: modulus, module, formal product of places, replete divisor,...

Also (in fact, normally) written \( m = \prod p^{n_p} \).
A word on open subgroups (II)

Let us define the following sets:
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If \( p \in \text{supp}(m) \) and it is non–archimedean, then

\[
W_m(p) = \left\{ \alpha \in k_p^* \mid \alpha \equiv 1 \mod p^n \right\} = 1 + p^n.
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W_m(p) = \{ \alpha \in k_p^* \mid \alpha \equiv 1 \mod p^n \} = 1 + p^n.
\]

If \( p \in \text{supp}(m) \) and it is archimedean, then

\[
W_m(p) = \mathbb{R}_+^*.
\]
Yet another word on open subgroups

Define now the subset:
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\[ I_m = \left( \prod_{p \notin \text{supp}(m)} k_p^* \times \prod_{p \in \text{supp}(m)} W_m(p) \right) \cap I_k. \]
Yet another word on open subgroups

Define now the subset:

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That is, \((\alpha)\) such that:

\[ \alpha_p \in k_p^* \quad \text{for all } p \]

\[ \alpha_p \in U_p \quad \text{for almost all } p \]

\[ \alpha_p \in W_m(p) \quad \text{for all } p \in \text{supp}(m) \]
Then we consider the sets

\[ W_m = \{ (\alpha_p) \in I_m \mid \alpha_p \in U_p, \text{ for all } p \text{ finite, } p \not\in \text{supp}(m) \} \]
A penultimate word on open subgroups

Then we consider the sets

\[ W_m = \{(\alpha_p) \in I_m \mid \alpha_p \in U_p, \text{ for all } p \text{ finite, } p \notin \text{supp}(m)\} \]

In other words:

\[ W_m = \prod_{\text{infinite } p \notin \text{supp}(m)} k_p^* \times \prod_{p \in \text{supp}(m)} W_m(p) \times \prod_{\text{finite } p \notin \text{supp}(m)} U_p \]
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Then we consider the sets

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That is, \((\alpha)\) such that

\[ \alpha_p \in k^*_p \quad \text{for all } p \text{ infinite} \]
\[ \alpha_p \in U_p \quad \text{for all } p \text{ finite} \]
\[ \alpha_p \in W_m(p) \quad \text{for all } p \in \text{supp}(m) \]
$W_m$ is called the congruence subgroup of $m$. 
A last word on open subgroups

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AND

Every open subgroup of $I_k$ must contain some congruence subgroup $W_m$. 
The canonical embedding

Let $K|k$ be a finite extension of number fields. Then we have a canonical embedding

$$\mathbb{A}_k \rightarrow \mathbb{A}_K$$

$$(\alpha_p) \mapsto (\alpha_{\mathfrak{p}}), \text{ where } \alpha_{\mathfrak{p}} = \alpha_p, \text{ whenever } \mathfrak{p} \mid p$$
The canonical embedding

Let $K|k$ be a finite extension of number fields. Then we have a canonical embedding

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$$(\alpha_p) \longmapsto (\alpha_{\mathfrak{P}}), \text{ where } \alpha_{\mathfrak{P}} = \alpha_p, \text{ whenever } \mathfrak{P}|p$$

It is, in fact, an injective homomorphism which induces also an embedding

$$\mathfrak{l}_k \hookrightarrow \mathfrak{l}_K.$$
The canonical embedding

Let $K|k$ be a finite extension of number fields. Then we have a canonical embedding

$$A_k \hookrightarrow A_K \quad (\alpha_p) \mapsto (\alpha_{\mathfrak{p}}), \text{ where } \alpha_{\mathfrak{p}} = \alpha_p, \text{ whenever } \mathfrak{p}|p$$

It is, in fact, an injective homomorphism which induces also an embedding $I_k \hookrightarrow I_K$.

Remark.– If $\mathfrak{p}, \mathfrak{p}'|p$ then for all $\alpha \in I_k$, $\alpha_{\mathfrak{p}} = \alpha_{\mathfrak{p}'}$ (criterion for being in $I_k$).
Idèles and field isomorphisms

Let $\sigma : L \rightarrow K$ be a field isomorphism, $\mathfrak{P}$ a place in $L$. 
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It induces an isomorphism (an isometry actually) $\sigma : L_{\mathfrak{P}} \rightarrow K_{\sigma \mathfrak{P}}$.

Idea: Take $\mathfrak{P}$–limits to $\sigma \mathfrak{P}$–limits.
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Let $\sigma : L \rightarrow K$ be a field isomorphism, $\mathfrak{p}$ a place in $L$.

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Idea: Take $\mathfrak{p}$–limits to $\sigma\mathfrak{p}$–limits.

As for idèles is concerned $\alpha$ goes to $\sigma \alpha$, where

$$\alpha \mathfrak{p} \in L_{\mathfrak{p}} \implies (\sigma \alpha)_{\sigma \mathfrak{p}} = \sigma (\alpha \mathfrak{p}) \in K_{\sigma \mathfrak{p}}.$$
Let $K|k$ be a Galois extension with Galois group $G$. 
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$\sigma \in G$ is an automorphism of $K$, therefore induces an automorphism $\sigma : I_K \rightarrow I_K$,

making $I_K$ a $G$–module.
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$\sigma \in G$ is an automorphism of $K$, therefore induces an automorphism

$$\sigma : I_K \rightarrow I_K,$$

making $I_K$ a $G$–module.

As usual:

$$I^K_G = \{ \alpha \in I_K \mid \sigma \alpha = \alpha, \ \forall \sigma \in G \}.$$
Galois descent for idèles (II)

Theorem. \(- I_K^G = I_k. \)
Theorem.– $I_K^G = I_k$. 

If $\alpha = (\alpha_\mathfrak{p}) \in I_K^G$, then $\alpha_{\sigma\mathfrak{p}} = (\sigma\alpha)_{\sigma\mathfrak{p}}$. 
Galois descent for idèles (II)

**Theorem.** $I^G_K = I_k$.

If $\alpha = (\alpha \mathfrak{p}) \in I^G_K$, then $\alpha_{\sigma \mathfrak{p}} = (\sigma \alpha)_{\sigma \mathfrak{p}}$.

Then, if we take $\sigma \in \text{Gal}(K \mathfrak{p} | k_p)$, $\sigma \mathfrak{p} = \mathfrak{p}$, and then $\alpha \mathfrak{p} \in k^*_p$.

For an arbitrary $\sigma$, it takes $\mathfrak{p}$ into $\mathfrak{p}'$ which also divides $p$. Then $\alpha$ begin fixed implies $\alpha \mathfrak{p} = \alpha \mathfrak{p}'$, hence $\alpha \in I_k$. 
Norm (revisited)

Take $\alpha \in I_K$, $\mathfrak{p} \in P_K$ (a place in $K$).

Multiplication by $\alpha \mathfrak{p}$ is a $k_p$-linear automorphism of $K_{\mathfrak{p}}$, and its determinant is set to be

$$N_{K_{\mathfrak{p}}|k_p}(\alpha \mathfrak{p}) \in k_p.$$
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Multiplication by \( \alpha \mathfrak{p} \) is a \( k_p \)-linear automorphism of \( K_{\mathfrak{p}} \), and its determinant is set to be

\[
N_{K_{\mathfrak{p}}|k_p} (\alpha \mathfrak{p}) \in k_p.
\]

In fact, these local norms induce a global norm \( I_K \rightarrow I_k \).

Let \( \alpha \in I_K \), then

\[
N_{K|k} (\alpha)_p = \prod_{\mathfrak{p}|p} N_{K_{\mathfrak{p}}|k_p} (\alpha \mathfrak{p})
\]
Norm (revisited)

Take $\alpha \in \mathcal{I}_K$, $\mathfrak{p} \in \mathcal{P}_K$ (a place in $K$).

Multiplication by $\alpha \mathfrak{p}$ is a $k_p$–linear automorphism of $K_{\mathfrak{p}}$, and its determinant is set to be

$$N_{K_{\mathfrak{p}}|k_p}(\alpha \mathfrak{p}) \in k_p.$$ 

In fact, these local norms induce a global norm $\mathcal{I}_K \longrightarrow \mathcal{I}_k$.

Let $\alpha \in \mathcal{I}_K$, then

$$N_{K|k}(\alpha)_{p} = \prod_{\mathfrak{p}|p} N_{K_{\mathfrak{p}}|k_p}(\alpha \mathfrak{p}).$$

**Proposition.** – The set $N_{K|k}\mathcal{I}_K$ is an open and closed subgroup of $\mathcal{I}_k$. 
The embedding $I_k \hookrightarrow I_K$ takes clearly principal idèles into principal idèles.
The embedding of the idèle class groups

The embedding $I_k \hookrightarrow I_K$ takes clearly principal idèles into principal idèles.

**Proposition.** If $K|k$ is finite, then $I_k \hookrightarrow I_K$ induces an injection

$$C_k \hookrightarrow C_K.$$
The embedding $I_k \hookrightarrow I_K$ takes clearly principal idèles into principal idèles.

**Proposition.**— If $K|k$ is finite, then $I_k \hookrightarrow I_K$ induces an injection $C_k \hookrightarrow C_K$.

Mind that injectivity requires proving $I_k \cap K^* = k^*$, which is not very difficult taking $L|k$ a Galois extension such that $k \subset K \subset L$. 
**Remark.**– If $x \in K^*$, then $N_{K|k}(x)$ has the same meaning as idèle in $I_k$ and as element of $k^*$ (therefore as idèle).
Remark.– If \( x \in K^\ast \), then \( N_{K|k}(x) \) has the same meaning as idèle in \( I_k \) and as element of \( k^\ast \) (therefore as idèle).

Proposition.– The norm \( N_{K|k} \) induces a norm map

\[
N_{K|k} : C_K \longrightarrow C_k.
\]
**Remark.**– If \( x \in K^* \), then \( N_{K|k}(x) \) has the same meaning as idèle in \( I_k \) and as element of \( k^* \) (therefore as idèle).

**Proposition.**– The norm \( N_{K|k} \) induces a norm map

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\]

In fact, the set \( N_{K|k} C_K \) is an open and closed subgroup of \( C_k \) (easy from the idèle case).
Galois descent for idèле class group

**Proposition.** Let $K|k$ be Galois, $G$ its Galois group. Then $C_K$ is a $G$–module and $C^G_K = C_k$.
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**Proposition.** Let $K|k$ be Galois, $G$ its Galois group. Then $C_K$ is a $G$–module and $C^G_K = C_k$.

We begin with the following exact sequence

$$1 \rightarrow K^* \rightarrow I_K \rightarrow C_K \rightarrow 1$$

Then take $G$–fixed elements

$$1 \rightarrow (K^*)^G \rightarrow I_K^G \rightarrow C_K^G \rightarrow H^1(G, K^*)$$
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And then, by Hilbert–Noether’s Theorem 90,

$$1 \rightarrow k^* \rightarrow I_k \rightarrow C_K^G \rightarrow 1.$$
Here comes the cohomology!
Here comes the cohomology!

End of Part I
Here comes the cohomology!

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Coffee? Anyone?
The set-up

Let $M$ be a $G$–module (think of $G$ a Galois group, $M$ a number field).
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$$N_G(m) = \prod_{g \in G} g(m), \quad m \in M.$$
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Define the $G$–norm:

$$N_G(m) = \prod_{g \in G} g(m), \quad m \in M.$$ 

And consider the groups

$$M^G = \{ m \in M \mid g(m) = m, \forall g \in G \}$$

$$l_G(M) = \langle g(m) \cdot m^{-1} \mid m \in M, g \in G \rangle$$
The Tate cohomology

We define (actually, Tate did) the Tate cohomology groups as:
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\[ H^r_T(G, M) = \begin{cases} 
H^r(G, M) & \text{for } r > 0 \\
M^G/N_G(M) & \text{for } r = 0 \\
\ker(N_G)/I_G(M) & \text{for } r = -1 \\
H_{-r-1}(G, M) & \text{for } r < -1 
\end{cases} \]
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\end{cases} \]

It fits together homology and cohomology groups, via the induced homomorphism

\[ N_G : H_0 = M/I_G(M) \longrightarrow M^G \]
The Tate long sequence

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There is a long exact sequence

\[
\cdots \rightarrow H^{i-1}_T(G, M''') \rightarrow H^i_T(G, M') \rightarrow H^i_T(G, M) \rightarrow \\
\rightarrow H^i_T(G, M'') \rightarrow H^{i+1}_T(G, M') \rightarrow \cdots
\]
The Herbrand quotient (I)

When $G$ is a cyclic group, we have a special feature.
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**Proposition.** If $G$ is cyclic and finite, then

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The Herbrand quotient (I)

When $G$ is a cyclic group, we have a special feature.

**Proposition.**– If $G$ is cyclic and finite, then

$$H^i_T(G, M) \simeq H^{i+2}_T(G, M).$$

Let $1 \to M' \to M \to M'' \to 1$ be a short exact sequence.

Then the following diagram is exact

$$
\begin{array}{ccc}
H^{-1}_T(G, M') & \longrightarrow & H^{-1}_T(G, M) \\
\uparrow & & \downarrow \\
H^0_T(G, M'') & \longleftrightarrow & H^0_T(G, M)
\end{array}
$$

$$
\begin{array}{ccc}
& & H^{-1}_T(G, M''') \\
& & \downarrow \\
& & H^0_T(G, M')
\end{array}
$$
The Herbrand quotient (II)

When the groups $H^i_T(G, M)$ are finite, we define the Herbrand quotient as

$$h(M) = \frac{#H^0_T(G, M)}{#H^{-1}_T(G, M)}.$$
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Some remarks:

1) If two of $M, M', M''$ have Herbrand quotient, so does the third.

2) If $M$ is finite, then $h(M) = 1$.

3) $h(M)$ is usually much more easy to compute than the actual Tate groups.
The Goal

In the previous set up, let us consider $K|k$ a Galois extension with Galois group $G$, $M = C_K$.

Here $N_G = N_{K|k}$, as we (quickly) mentioned.
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Moreover,

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H^0 = \frac{C_K^G}{N_G(C_K)} = \frac{C_k}{N_{K|k}C_K}.
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The First Inequality

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In the previous set up, let us consider $K|k$ a Galois extension with Galois group $G$, $M = C_K$.

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We want to prove

$$[C_k : N_{K|k}C_K] = [K : k]$$

and, to begin with, we will see

$$[C_k : N_{K|k}C_K] \geq [K : k]$$
Local $\rightarrow$ global works!

Let $S$ be finite, $S_\infty \subset S \subset P_k$.

\[ \bar{S} = \{ \mathfrak{p} \in P_K \text{ above places in } S \} \]
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Let $S$ be finite, $S_\infty \subset S \subset P_k$.

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Lazy notation: $I_K^S = I_K^{\bar{S}}$. 
Local $\rightarrow$ global works!

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\bar{S} = \{ \mathfrak{P} \in P_K \text{ above places in } S \}
$$

Lazy notation: $I^S_K = I^\bar{S}_K$.

**Proposition.**– If $K|k$ cyclic, and $S$ contains all ramified primes, for $i = 1, 2$:

$$
H^i \left( G, I^S_K \right) = \bigoplus_{p \in S} H^i \left( G_{\mathfrak{P}}, K_{\mathfrak{P}}^* \right)
$$

$$
H^i \left( G, I_K \right) = \bigoplus_{p} H^i \left( G_{\mathfrak{P}}, K_{\mathfrak{P}}^* \right)
$$

where $G_{\mathfrak{P}}$ is the Galois group of $K_{\mathfrak{P}}|k_p$ and $\mathfrak{P}|p$.
First brick

The fact that the global Tate cohomology can be decomposed and recovered from local pieces has two interesting corollaries.
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**Proposition.**— Let $K|k$ be a cyclic extension, $\alpha \in I_k$. Then $\alpha \in N_{K|k} C_K$ if and only if $\alpha_p \in N_{K_P|k_P} K_P$, for all $P|p$. 


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**Proposition.**– If $K|k$ is cyclic, and $S$ contains all ramified primes,

$$h \left( G, I^S_K \right) = \prod_{p \in S} n_p,$$

where $n_p = [K_P : k_p]$. 
For $S$ finite, $S_\infty \subset S \subset P_k$, let

$$K^S = K \cap I_K^S,$$

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**Proposition.**– Assume $K|k$ is cyclic. Then

$$h \left( G, K^S \right) = \frac{1}{[K : k]} \prod_{p \in S} n_p,$$

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where $n_p = [K_p : k_p]$.

The proof is based on local considerations, plus some (pretty technical) work on lattices.
Statement of the First Inequality

**Theorem (First Inequality).**— Let $K|k$ be cyclic, with Galois group $G$. 
The First Inequality

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Theorem (First Inequality).— Let $K|k$ be cyclic, with Galois group $G$. Then

$$h(G, C_K) = \frac{\#H^0(G, C_K)}{\#H^{-1}(G, C_K)} = [K : k]$$

In particular,

$$[C_k : N_{K|k}C_K] \geq [K : k]$$
Proof of the First Inequality

Take, as previously, \( S \subset P_k \) a set of places such that

- \( S_\infty \subset S \).

- \( S \) contains all primes that split in \( K \) such that \( I_K^S \cdot K^* = I_K \).
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$$1 \to K^S \to I_K^S \to (I_K^S \cdot K^*)/K^* \cong C_K \to 1$$
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Then

$$h(G, C_K) = \frac{h(G, I^S_K)}{h(G, K^S)} = [K : k]$$
Corollaries of the First Inequality

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**Corollary 1.** Assume $K|k$ is cyclic of order $p^\nu$, $p$ prime. Then there are infinitely many places in $P_k$ that do not split.
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**Corollary 1.**—Assume $K|k$ is cyclic of order $p^\nu$, $p$ prime. Then there are infinitely many places in $P_k$ that do not split.

**Corollary 2.**—Assume $K|k$ is finite. If almost all primes of $k$ split completely in $K$, then $k = K$. 
What are we proving (sort of)

Theorem.— Let $K|k$ be a Galois extension with Galois group $G$. Then:
1) $C_k/N_{K|k}C_K$ is finite, and its order divides $[K:k]$.
2) $H^1(G, C_K) = 1$.
3) $H^2(G, C_K)$ is finite, of order at most $[K:k]$.
The Second Inequality

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The First Inequality implies that, if $G$ cyclic, all three are equivalent.

Furthermore, in that case, $H^2(G, C_K)$ has order $[K : k]$. 
Overview of an analytic proof

There is an analytic proof of this theorem, and it is shorter, but the techniques involved are rather different, so we will only sketch it.
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Take $K|k$ finite, $L|k$ its Galois closure. Set

$$S = \{ \text{primes of } k \text{ that split completely in } K \}$$

The set $S$ has Dirichlet density $1/[L : k]$ (a special case of Chebotarev).
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$$S = \{ \text{primes of } k \text{ that split completely in } K \}$$

The set $S$ has Dirichlet density $1/[L : k]$ (a special case of Chebotarev).

Via $L$–series and Fourier analysis this is related to a set

$$I_m/(P_m N_{L|k} J_m)$$

which is an ideal version of $C_k/N_{K|k} C_K$, and has the same number of elements.
The reduction

The algebraic proof of our theorem relies at first on two very important reductions:
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1) It is enough to consider the case where \( K \mid k \) is cyclic of prime order (We move from \( G \) to all of its Sylow \( p \)–subgroups, and prove that suffices)
The algebraic proof of our theorem relies at first on two very important reductions:

1) It is enough to consider the case where $K|k$ is cyclic of prime order
   (We move from $G$ to all of its Sylow $p$–subgroups, and prove that suffices)

2) It is enough to consider the case where $k$ contains a $p$-th root of unity.
   (If not, we add $\zeta_p$ and prove, by diagram chasing, that things do not change a lot)
The key case: Set–up

Assume $K|k$ is cyclic of order $p$ and $k$ contains a $p$–th root of unity $\zeta_p$. 
The key case: Set–up

Assume \( K|k \) is cyclic of order \( p \) and \( k \) contains a \( p \)–th root of unity \( \zeta_p \).

Let \( S \subset P_k \) be a finite set such that:

1) \( S_\infty \subset S \).

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3) \( I_k = I_k^S \cdot k^* \).
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1) $S_\infty \subset S$.
2) The primes that split in $K$ are also in $S$.
3) $I_k = I^S_k \cdot k^*$.

And write, $k^S = I^S_k \cap k^*$, $s = \#S$. 
The key case: Auxiliary places

We want to construct a subgroup of $C_k$, of index $[K : k]$ which consists of norms from $C_K$. 
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Let us take $T \subset P_k$, finite, such that $T \cap S = \emptyset$, and set

$$J = \prod_{p \in S} (k_p^*)^p \times \prod_{p \in T} k_p^* \times \prod_{p \notin S \cup T} U_p.$$
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Let us also define $\Delta = (K^*)^p \cap k^S$. 
The key case: Three steps

With these notations, we can prove (with some work):

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1) $K = k \left( p^{\sqrt{\Delta}} \right)$.

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$$\Delta = \ker \left( k^S \xrightarrow[]{} \prod_{p \in T} k_p^*/(k_p^*)^p \right)$$
The key case: Three steps

With these notations, we can prove (with some work):

1) \( K = k \left( \sqrt[p]{\Delta} \right) \).

2) There exists \( T \) such that \( \# T = s - 1 \) and

\[
\Delta = \ker \left( k^S \longrightarrow \prod_{p \in T} k_p^* / (k_p^*)^p \right)
\]

3) For such a \( T \), set \( C_k^{S, T} = (J \cdot K^*)/K^* \). Then

\[
\left[ C_k / C_k^{S, T} \right] = [K : k] = p,
\]

and \( C_k^{S, T} \subset N_{K|k} C_K \).
The Second Inequality and the Class Field Axiom

The construction of $T$ finishes the proof of the Second Inequality

$$[C_k : N_{K|k}C_K] \leq p.$$
The Second Inequality and the Class Field Axiom

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In Neukirch’s terminology, we have

**Theorem (The Global Class Field Axiom).**— Let $K|k$ be cyclic. Then

$$\#H^i(G, C_K) = \begin{cases} 
[K : k] & i = 0 \\
1 & i = -1 
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The Second Inequality and the Class Field Axiom

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$$\# H^i (G, C_K) = \begin{cases} [K : k] & i = 0 \\ 1 & i = -1 \end{cases}$$

**Remark.**— We knew that $x$ as principal idèle is a norm if and only if it is a norm locally everywhere, but it does *not* necessarily have to be the norm of a principal idèle if $K|k$ is not cyclic.
Corollary (Hasse’s Norm Theorem).— Let $K|k$ be a cyclic extension. Then $x \in k^*$ is the norm of an element of $K^*$ if and only if $x$ is a norm in every $K_{\mathbb{Q}}|k_p$. 
Corollary (Hasse’s Norm Theorem).— Let \( K \mid k \) be a cyclic extension. Then \( x \in k^* \) is the norm of an element of \( K^* \) if and only if \( x \) is a norm in every \( K_{\mathfrak{p}} \mid k_{\mathfrak{p}} \).

From

\[
1 \to K^* \to \mathcal{I}_K \to C_K \to 1
\]

we get

\[
1 = H^{-1}(G, C_K) \to H^0(G, K^*) \to H^0(G, \mathcal{I}_K)
\]
Hasse’s Norm Theorem

**Corollary (Hasse’s Norm Theorem).**— Let \( K \mid k \) be a cyclic extension. Then \( x \in k^* \) is the norm of an element of \( K^* \) if and only if \( x \) is a norm in every \( K_{\mathfrak{p}} \mid k_p \).

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we get

\[
1 = H^{-1}(G, C_K) \rightarrow H^0(G, K^*) \rightarrow H^0(G, I_K)
\]

Therefore

\[
H^0(G, K^*) = k^*/N_{K \mid k}K^* \hookrightarrow H^0(G, I_K) = \bigoplus_p H^0(G_{\mathfrak{p}}, K_{\mathfrak{p}}^*),
\]

which is the statement of the theorem, in a sophisticated way.
Let $K|k$ be abelian, with Galois group $G$. Let $p \in P_k$, $\mathfrak{P} \in P_K$ such that $\mathfrak{P}|p$. 
Artin Reciprocity Law: The local map

Let $K|k$ be abelian, with Galois group $G$. Let $p \in P_k$, $\mathfrak{P} \in P_K$ such that $\mathfrak{P}|p$.

We recall this set from LCFT

$$D(\mathfrak{P}) = \{ \sigma \in G \mid \sigma \mathfrak{P} = \mathfrak{P} \} \cong \text{Gal}(K_{\mathfrak{P}}|k_p).$$
Artin Reciprocity Law: The local map

Let $K|k$ be abelian, with Galois group $G$. Let $p \in P_k$, $\mathfrak{P} \in P_K$ such that $\mathfrak{P}|p$.

We recall this set from LCFT

$$D(\mathfrak{P}) = \{\sigma \in G \mid \sigma \mathfrak{P} = \mathfrak{P}\} \cong \text{Gal} (K\mathfrak{P}|k_p).$$

The local Artin map is

$$\phi_p : k_p \longrightarrow D(\mathfrak{P}) \subset G.$$
Artin Reciprocity Law: Patching local maps

We can fit together local Artin maps by means of the following result.
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**Proposition.**— There exists a unique homeomorphism

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such that, for all \( K \subset k^{ab} \) finite, and every \( p \in P_k, \mathfrak{p} \in P_K \) with \( \mathfrak{p} \mid p \), the following diagram
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\[
\begin{array}{ccc}
  k_p^* & \xrightarrow{\phi_p} & D(\mathfrak{P}) \cong \text{Gal} \left( K_{\mathfrak{P}} | k_p \right) \\
  \downarrow & & \downarrow \\
  I_k & \xrightarrow{\phi_k} & G \\
  \alpha & \longmapsto & \phi_k(\alpha) |_K
\end{array}
\]

commutes.
Artin Reciprocity Law: How to patch

The definition of $\phi_k$ can be seen as follows. Take $\alpha \in \mathfrak{I}_k$, and $K \subset K^{ab}$ such that $K|k$ is finite. Then:

1) $\phi_p(\alpha_p) = 1$ except for finitely many $p$ (it is 1 when $\alpha_p \in U_p$ and $K_{\mathfrak{P}}|k_p$ is unramified).

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Artin Reciprocity Law: How to patch

The definition of $\phi_k$ can be seen as follows. Take $\alpha \in \mathbb{L}_k$, and $K \subset K^{ab}$ such that $K|k$ is finite. Then:

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2) The unique choice is then (for a fixed $K$)

$$\phi_{K|k}(\alpha) = \prod_p \phi_p (\alpha_p).$$

3) A field extension corresponds to a unique extension of $\phi_k$ (because of the local properties of the Artin maps).
Theorem (Artin Reciprocity Law).— The homeomorphism

$$\phi_k : I_k \longrightarrow \text{Gal} \left( k^{ab} | k \right)$$

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Artin Reciprocity Law: The statement

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verifies:

1) $\phi_k(k^*) = 1$.

2) For all abelian finite extensions $K|k$, $\phi_k$ induces an isomorphism:

$$\phi_{K|k} : I_k/(k^* \cdot N_{K|k} I_K) \rightarrow \text{Gal} (K|k)$$
Artin Reciprocity Law: In terms of $C_k$

Artin Reciprocity Law can be restated in terms of the idèle class group as follows:
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2) $\phi_k$ induces an isomorphism $\phi_{K|k} : C_k/N_{K|k}C_K \cong \text{Gal}(K|k)$
A brief recall from LCFT. Let us consider $K|k$.

Take $p \in P_k$ and $\mathcal{P} \in P_K$ such that $\mathcal{P}|p$ and $\mathcal{P}$ is unramified over $p$. 
Artin Reciprocity Law: Frobenius elements

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We have

$$\text{Gal } (\mathcal{O}_K/\mathfrak{p} : \mathcal{O}_k/p) \simeq \text{Gal } (K_{\mathfrak{p}}|k_p)$$

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The Frobenius element $(\mathfrak{p}, K|k)$ is the element of $D(\mathfrak{p})$ corresponding to the Frobenius element.
Artin Reciprocity Law: Properties of the Frobenius element

The Frobenius element can alternatively be described as the only element \( \sigma \in G \) such that:

1) \( \sigma \mathfrak{P} = \mathfrak{P} \).
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1) $\sigma \mathfrak{P} = \mathfrak{P}$.

2) For all $\alpha \in \mathcal{O}_k$, $\sigma \alpha = \alpha^q \mod \mathfrak{P}$, where $q = \#(\mathcal{O}_k/p)$. 

J.M. Tornero (Universidad de Sevilla)
Artin Reciprocity Law: Properties of the Frobenius element

The Frobenius element can alternatively be described as the only element \( \sigma \in G \) such that:

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Another interesting property is that, as \( G \) acts transitively on the set of primes dividing \( p \),

\[ \{(\mathfrak{p}, K|k) \mid \mathfrak{p}|p\} \]

is a conjugacy class in \( G \), noted \((p, K|k)\).

The Frobenius elements \((\mathfrak{p}, K|k)\), for the primes \( \mathfrak{p} \) which do not ramify generate the Galois group of \( K|k \).
To prove Artin Reciprocity Law it suffices proving:
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**Key Theorem.**– Let $K|k$ be a finite abelian extension with Galois group $G$. Then $\phi_{K|k} : I_K \rightarrow G$ is trivial on the principal idèles.
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**Key Theorem.** Let $K|k$ be a finite abelian extension with Galois group $G$. Then $\phi_{K|k} : I_K \rightarrow G$ is trivial on the principal idèles.

It is the most delicate part of the proof because:

1) The norm subgroup $N_{K|k} I_K$ is contained in the kernel of $\phi_{K|k}$ because it is locally.
2) Once we assume the Key Theorem, we have a homomorphism

\[ \mathfrak{l}_k / (k^* \cdot N_{K|k} \mathfrak{l}_K) \to \text{Gal}(K|k) \]

which is surjective because we can explicitly construct an idèle \( \alpha \) such that \( \phi_{K|k}(\alpha) \) is a Frobenius element for an unramified prime, and these elements generate \( G \).
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3) From the Second Inequality

\[ [I_k : k^* \cdot N_{K|k}I_K] \leq [K : k]. \]
Artin Reciprocity Law: Proof (II)

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\[ [I_k : k^* \cdot N_{K|k} I_K] \leq [K : k]. \]

Then 2) and 3) (together with the Key Theorem) prove Artin Reciprocity Law.
Artin Reciprocity Law: Strategy for the Key Theorem

We will not go into detail, but the steps to prove the Key Theorem are:

1) Prove that, if it works for $K|k$, it works for any subextension, and also for joint extensions (given $K'|k$, considering $K' \cdot K|K'$) (technical, not difficult).
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1) Prove that, if it works for $K|k$, it works for any subextension, and also for joint extensions (given $K'|k$, considering $K' \cdot K|K'$) (technical, not difficult).

2) Prove that it suffices to consider cyclic cyclotomic extensions (complicated).
The BIG results

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1) Prove that, if it works for $K|k$, it works for any subextension, and also for joint extensions (given $K'|k$, considering $K' \cdot K|K'$) (technical, not difficult).

2) Prove that it suffices to consider cyclic cyclotomic extensions (complicated).

3) Check that it is true for cyclotomic extensions (fairly easy).
We finish our presentation with a outstanding result, based on Artin Reciprocity Law.

**Theorem (Existence Theorem).**— Let $k$ be a number field. The finite abelian extensions $K|k$ are in one–to–one correspondence with the open subgroups of $C_k$ of finite index

$$K \longleftrightarrow N_{K|k} C_K$$
The Existence Theorem

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Thanks to the Reciprocity Law, it suffices to prove that every subgroup of finite index contains a norm subgroup.

**Definition.**– The field corresponding to an open subgroup $N \subset C_k$ is called the *class field* of $N$. 

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idèles 

October 2009 

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Thanks a lot!
Thanks a lot!

Any questions?
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Well, thanks again, you’ve been a wonderful audience!