# Definitions

1. A *reductive group* over $F$ is an affine algebraic group $\boldsymbol{G}$ defined over $F$, with trivial unipotent radical.

   e.g. $\boldsymbol{GL}_n$, $\boldsymbol{SL}_n$, $\boldsymbol{Sp}_{2n}$, ...

2. $\boldsymbol{G}$ is ($F$-)split if it has an $F$-split maximal torus.

   e.g. $\boldsymbol{GL}_n$ is $F$-split.

   The group $\boldsymbol{G}(F) = E^1$, for $E/F$ quadratic, is not split.

3. $\boldsymbol{G}$ is *quasi=split* it it has a Borel subgroup defined over $F$.

If $\boldsymbol{G}$ is split then it is quasi-split.

# Example: $\boldsymbol{U}(2,1)(E/F)$

$E/F$ unramified quadratic, $\mathrm{Gal}(E/F)=\{1,\sigma\}$, $p\neq 2$, and

$$w_0 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

$$\begin{aligned} G &= \boldsymbol{U}(2,1)(E/F) \\ &= \boldsymbol{G}(F) \;=\; \{g\in \boldsymbol{GL}_3(E) : w_0{}^{\sigma}g^T w_o g = 1\}. \end{aligned}$$

$\boldsymbol{G}$ is defined over $F$ and has compact centre

$$Z \;=\; \boldsymbol{Z}(F) \;=\; \left\{ \begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix} : x\in E^1 \right\}.$$

$\boldsymbol{G}$ is isomorphic to $\boldsymbol{GL}_3$ *over* $E$ (but not over $F$).

# Example: $\boldsymbol{U}(2,1)(E/F)$

**Relative** (over F)

A maximal $F$-split torus is

$$A$$

$$\left\{\begin{pmatrix}\lambda & & \\ & 1 & \\ & & \lambda^{-1}\end{pmatrix} : \lambda \in F^\times\right\}$$

$$N_G(A) \;=\; T \cup w_0 T.$$

Relative Weyl group

$$W_F \;=\; N_G(A)/Z_G(A) \;\simeq\; S_2.$$

**Absolute** (over $\bar{F}$ or $E$)

An ($E$-split) maximal torus is

$$T = \boldsymbol{T}(F) = Z_G(A)$$

$$\left\{\begin{pmatrix}x & & \\ & y & \\ & & {}^{\sigma}x^{-1}\end{pmatrix} : \begin{matrix} x \in E^\times \\ y \in E^1 \end{matrix}\right\}$$

Absolute Weyl group

$$W \simeq \; S_3.$$

# Example: $\boldsymbol{U}(2,1)(E/F)$

A minimal parabolic subgroup over $F$ is

$$B = \boldsymbol{B}(F) = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \in G \right\},$$

which is a Borel subgroup, so $\boldsymbol{G}$ is quasi-split.

We have a relative Bruhat decomposition $G = \bigcup_{w \in W_F} B\dot{w}B$.

The standard parabolic subgroups defined over $F$ are in bijection with the subsets of a (suitable) set of generators for $W_F$. Here the only proper parabolic subgroups defined over $F$ are the Borel subgroups.

# Example: $\boldsymbol{U}(2,1)(E/F)$

We can write the unipotent radical

$$U = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & {}^{\sigma}x \\ & & 1 \end{pmatrix} : \begin{array}{l} x, y \in E \\ y + {}^{\sigma}y + x^{\sigma}x = 0 \end{array} \right\} = U_{\alpha} U_{2\alpha},$$

where

$$U_{\alpha} = \left\{ \begin{pmatrix} 1 & x & -\frac{x^{\sigma}x}{2} \\ & 1 & {}^{\sigma}x \\ & & 1 \end{pmatrix} : x \in E \right\}$$

$$U_{2\alpha} = \left\{ \begin{pmatrix} 1 & & y \\ & 1 & \\ & & 1 \end{pmatrix} : \begin{array}{l} y \in E \\ y + {}^{\sigma}y = 0 \end{array} \right\}$$

Both are preserved under the conjugation action of $A$.

# Example: $\boldsymbol{U}(2,1)(E/F)$

There are two non-conjugate maximal compact subgroups of $G$:

1. $K_1 \;=\; \boldsymbol{GL}_3(\mathfrak{o}_E) \cap G$, the "nice" one;

   $K_1/K_1^1 \simeq \boldsymbol{U}(2,1)(k_E/k_F)$.

2. $K_2 \;=\; \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E \end{pmatrix} \cap G$, the "not-so-nice" one;

   $K_2/K_2^1 \simeq \boldsymbol{U}(1)(k_E/k_F) \times \boldsymbol{U}(1,1)(k_E/k_F)$.

There is also an Iwahori subgroup (the only non-maximal parahoric subgroup)

$$\mathcal{I} \;=\; K_1 \cap K_2 \;=\; \begin{pmatrix} \mathfrak{o}_E^\times & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{o}_E^\times & \mathfrak{o}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{o}_E^\times \end{pmatrix} \cap G.$$

# Example: $\boldsymbol{U}(2,1)(E/F)$

Putting

$$Z_G(A)^0 \;=\; \mathcal{I}\cap N_G(A) \;=\; \begin{pmatrix} \mathfrak{o}_E^\times & & \\ & \mathfrak{o}_E^\times & \\ & & \mathfrak{o}_E^\times \end{pmatrix} \cap G,$$

we have an affine Weyl group $\widetilde{W} \;=\; N_G(A)/Z_G(A)^0$, which is an infinite dihedral group, with generators

$$w_0 \;=\; \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \qquad w_1 \;=\; \begin{pmatrix} & & \varpi_F^{-1} \\ & 1 & \\ \varpi_F & & \end{pmatrix}.$$

Then $G = \bigcup_{w\in\widetilde{W}} \mathcal{I}\dot{w}\mathcal{I}$ (affine Bruhat decomposition).

# Appendix: Root data

Put $X(\boldsymbol{G}) = \{ \text{characters } \phi : \boldsymbol{G} \to \boldsymbol{G}_m \}$, and

$X_*(\boldsymbol{G}) = \{ \text{cocharacters } \theta : \boldsymbol{G}_m \to \boldsymbol{G} \}$.

e.g. $X(\boldsymbol{G}_m) \simeq \mathbb{Z}$, with $a \in \mathbb{Z}$ corresponding to $x \mapsto x^a$.

Then there is a pairing $X(\boldsymbol{G}) \times X_*(\boldsymbol{G}) \to \mathbb{Z}$ given by

$$\langle \phi, \theta \rangle = \chi \circ \theta \qquad (\text{in } X(\boldsymbol{G}_m) = \mathbb{Z}).$$

# $G = GL_n$

We have $X(\boldsymbol{T}) \simeq \mathbb{Z}^n$, with $\phi \in \mathbb{Z}^n$ corresponding to $t \mapsto t^\phi$,

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{\phi_1} \cdots t_n^{\phi_n};$$

and $X_*(\boldsymbol{T}) \simeq \mathbb{Z}^n$, with $\theta \in \mathbb{Z}^n$ corresponding to $x \mapsto x^\theta$.

$$x \mapsto \begin{pmatrix} x^{\theta_1} & & \\ & \ddots & \\ & & x^{\theta_n} \end{pmatrix}.$$

Then $\langle \phi, \theta \rangle = \sum_{i=1}^n \phi_i \theta_i$.

# $G = GL_n$

Put $\mathfrak{g} = \boldsymbol{M}_n(F)$, the Lie algebra of $G$.

$G$ acts of $\mathfrak{g}$ via the adjoint action

$$Ad(g)y \;=\; gyg^{-1}, \qquad \text{for } g \in G, y \in \mathfrak{g}.$$

$Ad(T)$ is a set of commuting semisimple automorphisms of $\mathfrak{g}$ so is diagonalizable – i.e. we can find a basis of simultaneous eigenvectors.

A simultaneous eigenvalue is a character $\phi \in X(\boldsymbol{T})$, with eigenspace

$$\mathfrak{g}_\phi \;=\; \{y \in \mathfrak{g} : \mathrm{Ad}\,(t)y = t^\phi y\ \forall t \in T\}.$$

# $G = GL_n$

If $\phi = 0$ (trivial character)

$$\mathfrak{g}_0 = \{\text{diagonal matrices}\} = \mathfrak{t}.$$

The set of *roots* of $G$ (relative to $T$) is

$$\begin{aligned} \Phi &= \{\phi \neq 0 : \mathfrak{g}_\phi \neq 0\} \\ &= \{\alpha_{ij} : 1 \leq i, j \leq n,\ i \neq j\}, \end{aligned}$$

where $t^{\alpha_{ij}} = t_i t_j^{-1}$ and

$$\mathfrak{g}_{\alpha_{ij}} = \mathfrak{u}_{ij}.$$

Also define the set of *dual roots* in $X_*(\boldsymbol{T})$

$$\Phi^\vee = \{\alpha_{ij}^\vee : 1 \leq i, j \leq n,\ i \neq j\},$$

so that $\langle \alpha_{ij}, \alpha_{ij}^\vee \rangle = 2$.

# $G = GL_n$

The Weyl group $W = N_{\boldsymbol{G}}(\boldsymbol{T})/Z_{\boldsymbol{G}}(\boldsymbol{T})$ acts on $X(\boldsymbol{T})$ (and $X_*(\boldsymbol{T})$),

$$t^{w(\phi)} = \left(\dot{w}^{-1} t \dot{w}\right)^{\phi}, \qquad x^{w(\theta)} = \dot{w} x^{\theta} \dot{w}^{-1},$$

and permutes $\Phi$ (and $\Phi^\vee$).

For example, if $s_{ij} \in W$ corresponds to the transpostion $(ij)$ then

$$s_{ij}(\phi) = \phi - \langle \phi, \alpha_{ij}^\vee \rangle \alpha_{ij}.$$

Then $(X(\boldsymbol{T}), \Phi, X_*(\boldsymbol{T}), \Phi^\vee)$ is a *root datum*.

# Reductive groups

Reductive groups $\boldsymbol{G}$ over an algebraically closed field are classified in terms of their root data.

In particular, $(X_*(\boldsymbol{T}), \Phi^\vee, X(\boldsymbol{T}), \Phi)$ is also a root datum so corresponds to a (dual) reductive group also. Then

$$ {}^L G^0 = \{\textit{complex} \text{ points of the dual group}\}. $$

e.g. For $G = \boldsymbol{GL}_n$, $\quad {}^L G^0 = \boldsymbol{GL}_n(\mathbb{C})$,

$\boldsymbol{SL}_n$, $\quad {}^L G^0 = \boldsymbol{PGL}_n(\mathbb{C})$, ...

# Non-split groups

When $\boldsymbol{G}$ is not $F$-split then we have also the *relative roots* (the $\pm\alpha, \pm 2\alpha$ for $\boldsymbol{U}(2,1)(E/F)$) and the classification must also take into account the $F$-structure.

$\boldsymbol{G}$ splits over a finite Galois extension $E/F$ and then $\mathrm{Gal}(E/F)$ acts on the root datum of $\boldsymbol{G}$.

One deduces from this an action of $\mathrm{Gal}(E/F)$ on the complex dual group ${}^{L}G^{0}\ldots$