Irreducible representations

In order to understand $\mathrm{Irr}(G)$, we have three problems:

1. Construct irreducible supercuspidal representations (of Levi subgroups).
2. Find cuspidal subquotients of $\iota_{L,P}^{G}\,\sigma$, for σ supercuspidal.
3. Decompose $\iota_{L,P}^{G}\,\sigma$, for σ cuspidal.

Example: $G = \boldsymbol{GL}_2(F)$

If $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, then:

- $\iota_{T,B}^{G}(\chi_1 \otimes \chi_2)$ is irreducible unless $\chi_1\chi_2^{-1} = |\cdot|_F^{\pm 1}$;

Example: $G = \boldsymbol{GL}_2(F)$

If $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, then:

- if $\underline{\ell \nmid q_F^2 - 1}$ then we have

$$0 \to \chi|\cdot|_F^{1/2} \to \iota_{T,B}^G(\chi \otimes \chi|\cdot|_F) \to \chi|\cdot|_F^{1/2} St_G \to 0$$

and

$$0 \to \chi|\cdot|_F^{1/2} St_G \to \iota_{T,B}^G(\chi|\cdot|_F \otimes \chi) \to \chi|\cdot|_F^{1/2} \to 0;$$

Example: $G = \boldsymbol{GL}_2(F)$

If $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, then:

- if $\underline{q_F \equiv 1 \pmod{\ell} \text{ and } \ell \neq 2}$ then $|\cdot|_F = 1$ and

$$\iota_{T,B}^{G}(\chi \otimes \chi) = \chi \oplus \chi St_G;$$

Example: $G = \boldsymbol{GL_2}(F)$

If $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, then:

- if $\underline{q_F \equiv -1 \pmod{\ell}}$ then $|\cdot|_F = |\cdot|_F^{-1}$ and $\iota_{T,B}^G(\chi \otimes \chi|\cdot|_F)$ has length 3 with composition factors:
  - a 1-dimensional subrepresentation;
  - a 1-dimensional quotient;
  - a “special representation” which is cuspidal but not supercuspidal.

Supercuspidal representations

Supercuspidal representations are constructed by compact induction from compact-mod-centre subgroups: if

- $\widetilde{K}$ is an open compact-mod-centre subgroup of G,
- $\widetilde{\rho} \in \mathrm{Irr}\,(\widetilde{K})$, and
- $\pi = \text{c-Ind}\,_{\widetilde{K}}^{G}\,\widetilde{\rho}$ is irreducible,

then π is Z-compact so cuspidal.

Example: $G = \boldsymbol{GL}_2(F)$ and $C = \mathbb{C}$

Take $K = \boldsymbol{GL}_2(\mathfrak{o}_F)$ so $K/K^1 \simeq \boldsymbol{GL}_2(k_F)$.

Let $\overline{\sigma}$ be an irreducible *cuspidal* representation of $\boldsymbol{GL}_2(k_F)$.

i.e. $\overline{\sigma}|\boldsymbol{U}(k_F)$ *does not contain the trivial representation.*

$\sigma = \mathrm{Inf}_{K/K^1}^{K}(\overline{\sigma})$ and $\widetilde{\sigma}$ any extension to $\widetilde{K} = ZK$.

Then c-Ind$_{\widetilde{K}}^{G}\,\widetilde{\sigma}$ is an irreducible supercuspidal "level 0" representation of G.

Example: $G = GL_2(F)$ and $C = \mathbb{C}$

Put $\beta = \begin{pmatrix} 0 & \varpi_F^{-2} \\ \varpi_F^{-1} & 0 \end{pmatrix}$, so $E = F[\beta]/F$ is ramified quadratic.

Put $\mathcal{I}^2 = \begin{pmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F \end{pmatrix}$ and define

$$\psi_\beta(1+x) = \psi_F \circ \mathrm{tr}\,(\beta x)\,, \quad \text{for } 1+x \in \mathcal{I}^2.$$

ψ_F an additive character of F, trivial on $\mathfrak{p}_F$ but non-trivial on $\mathfrak{o}_F$.

Example: $G = GL_2(F)$ and $C = \mathbb{C}$

Put $\beta = \begin{pmatrix} 0 & \varpi_F^{-2} \\ \varpi_F^{-1} & 0 \end{pmatrix}$, so $E = F[\beta]/F$ is ramified quadratic.

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$$\psi_\beta(1+x) = \psi_F \circ \mathrm{tr}\,(\beta x), \quad \text{for } 1+x \in \mathcal{I}^2.$$

Put $\widetilde{J} = E^\times \mathcal{I}^2$, a compact-mod-centre open subgroup of G, $\widetilde{\kappa}$ any extension of ψ_β to $\widetilde{J}$.

Then $\text{c-Ind}_{\widetilde{J}}^{G}\,\widetilde{\kappa}$ is an irreducible supercuspidal representation (of "positive level $3/2$").