Introduction to L-functions II: Automorphic L-functions

References:

- D. Bump, *Automorphic Forms and Representations*.

- J. Cogdell, *Notes on L-functions for GL(n)*

- S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-functions*.

First lecture:

Tate's thesis, which develop the theory of L-functions for Hecke characters (automorphic forms of GL(1)). These are degree 1 L-functions, and Tate's thesis gives an elegant proof that they are "nice".

Today: Higher degree L-functions, which are associated to automorphic forms of $GL(n)$ for general n.

Goals:

(i) Define the L-function $L(s, \pi)$ associated to an automorphic representation π.

(ii) Discuss ways of showing that $L(s, \pi)$ is "nice", following the praradigm of Tate's thesis.

The group $G = \mathbf{GL}(n)$ over F

$F =$ number field.

Some subgroups of G:

(i) $Z \cong \mathbb{G}_m =$ the center of G;

(ii) $B =$ Borel subgroup of upper triangular matrices $= T \cdot U$;

(iii) $T =$ maximal torus of of diagonal elements $\cong (\mathbb{G}_m)^n$;

(iv) $U =$ unipotent radical of $B =$ upper triangular unipotent matrices;

(v) For each finite v,

$K_v = \mathrm{GL}_n(\mathcal{O}_v) =$ maximal compact subgroup.

Automorphic Forms on G

An automorphic form on G is a function

$$f : G(F)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

satisfying some smoothness and finiteness conditions.

The space of such functions is denoted by $\mathcal{A}(G)$. The group $G(\mathbb{A})$ acts on $\mathcal{A}(G)$ by right translation:

$$(g \cdot f)(h) = f(hg).$$

An irreducible subquotient π of $\mathcal{A}(G)$ is an automorphic representation.

Cusp Forms

Let $P = M \cdot N$ be any parabolic subgroup of G. For example, P is a subgroup of block upper triangular matrices.

Given $f \in \mathcal{A}(G)$, one may consider its "constant term" along N:

$$f_N(g) := \int_{N(F)\backslash N(\mathbb{A})} f(ng)\, dn.$$

Definition: Say that f is a cusp form if $f_N = 0$ for all $P = M \cdot N$.

Let $\mathcal{A}_0(G)$ denote the subspace of cusp forms. It is a $G(\mathbb{A})$-submodule of $\mathcal{A}(G)$.

In fact, this submodule is semisimple:

$$\mathcal{A}_0(G) = \bigoplus_{\pi} m(\pi)\, \pi,$$

as π ranges over irreducible representations of $G(\mathbb{A})$. A basic result says that $m(\pi) = 0$ or 1.

Restricted Tensor Product

Proposition: An irreducible automorphic representation π has the form

$$\pi \cong \otimes'_v \pi_v$$

where

- π_v is an irreducible representation of $G(F_v)$;
- for almost all v, π_v is an unramified representation, i.e.

$$\pi_v^{K_v} \neq 0.$$

- $\otimes'_v$ denotes restricted tensor product relative to the a K_v-fixed vector for almost all v.

To an irreducible automorphic representation, we would like to associate

$$L(s, \pi) = \prod_v L(s, \pi_v).$$

So we should first address the local questions:

(i) How to associate $L(s, \pi_v)$ to any π_v?

or less ambitiously

(ii) How to associate $L(s, \pi_v)$ to an unramified π_v?

Unramified Representations

Recall that the set of irreducible unramified representations of $G(F_v)$ has been classified.

Theorem: There is a natural bijection between

$$\{ \text{ irreducible unramified reps of } \mathsf{GL}_n(F_v)\}$$

and

$$\{ \text{ unordered n-tuples of elements of } \mathbb{C}^\times\}$$

Elements of the 2nd set can be thought of as diagonal matrices

$$s_v = \mathsf{diag}(a_1, ..., a_n) \in \mathsf{GL}_n(\mathbb{C}),$$

taken up to conjugacy.

Noting that $\mathsf{GL}_n(\mathbb{C})$ is the Langlands dual group of $G = \mathsf{GL}(n)$, have:

Restatement: The unramified irred reps of $\mathsf{GL}_n(F_v)$ are in natural bijection with conjugacy classes of semisimple elements in the Langlands dual group

$$\widehat{G} = \mathsf{GL}_n(\mathbb{C}).$$

This is the unramified "local Langlands correspondence" for $\mathsf{GL}(n)$. For $n = 1$, it reduces to the "unramified local class field theory".

Observe that the conjugacy class of a diagonal matrix

$$s_v = diag(a_1, ..., a_n).$$

as above is determined by its characteristic polynomial

$$P_{s_v}(T) = (1 - a_1 T).....(1 - a_n T).$$

Standard L-factors for unramified representations

Given this, we make the following definition

Definition: If π_v is an unramified representation, the standard L-factor associated to π_v is:

$$\begin{aligned} & L(s, \pi_v) \\ = & \frac{1}{P_{s_v}(q^{-s})} \\ = & \frac{1}{\prod_v (1 - a_i q_v^{-s})} \\ = & \frac{1}{\det(1 - s_v q_v^{-s})}. \end{aligned}$$

Thus, $L(s, \pi_v)$ determines s_v and hence π_v.

This is the analog of defining

$$L(s, \chi_v) = 1/(1 - \chi(\varpi_v) q_v^{-s})$$

for unramified χ_v, so that it is compatible with local class field theory and local Artin L-factor.

What if π_v is not unramified?

Should we simply set $L(s, \pi_v) = 1$, like in $n = 1$ case?

We would like to consider a family of zeta integrals associated to π_v, whose GCD is equal to $L(s, \pi_v)$ when π_v is unramified. Then we would define $L(s, \pi_v)$ for general π_v as the GCD of this family of zeta integral.

Such an approach was carried out by Godement-Jacquet, generalizing the zeta integrals of Tate.

Matrix Coefficients

Fix the irreducible rep π_v and let $\pi_v^\vee$ denote the contragredient rep of π_v.

So there is a natural $G(F_v)$-invariant pairing

$$\langle -, - \rangle : \pi_v^\vee \otimes \pi_v \longrightarrow \mathbb{C}$$

This is an element of

$$\mathrm{Hom}_{G(F_v)}(\pi_v^\vee \otimes \pi_v, \mathbb{C}).$$

By Schur's lemma, this Hom space is 1-dimensional.

For fixed vectors

$$f_v \in \pi_v \text{ and } f_v^\vee \in \pi_v^\vee,$$

one can form a function on $G(F_v)$:

$$\Phi_{f_v, f_v^\vee} : g \mapsto \langle f_v^\vee, g f_v \rangle.$$

Such a function is called a matrix coefficient of π_v.

The map

$$f_v \otimes f_v^\vee \mapsto \Phi_{f_v, f_v^\vee}$$

gives a $G(F_v)$-equivariant embedding

$$\pi_v^\vee \otimes \pi_v \hookrightarrow C^\infty(G(F_v)).$$

If the image of this map is contained in the space of functions which are compactly supported modulo $Z(F_v)$, then π_v is said to be a **supercuspidal** representation.

Local Zeta Integrals of Godement-Jacquet

Suppress v from notations.

Given

- $f \in \pi$;
- $f^\vee \in \pi^\vee$;
- $\phi \in S(M_n(F))$, where
$$M_n(F) = \ n \times n \text{ matrices over } F,$$
and
$$S(M_n(F)) = \{\text{Schwarz-Bruhat functions}\}$$

we set

$$Z(s, \phi, f, f^\vee) = \int_{\mathsf{GL}_n(F)} \phi(g) \cdot \langle f^\vee, g \cdot f \rangle \cdot |\det(g)|^s dg.$$

Observe that when $n = 1$, this reduces to the local zeta integral of Tate.

Local Theorem:

(i) There is a c such that whenever $Re(s) > c$, the integral $Z(s, \phi, f, f^\vee)$ converges absolutely for all ϕ, f and $f^\vee$.

(ii) The zeta integral is given by a rational function in q^{-s} and thus has meromorphic continuation to $\mathbb{C}$.

(iii) When π is unramified, the function

$$Z(s + \frac{n-1}{2}, \phi, f, f^\vee)/L(s, \pi)$$

is entire. Moreover, if ϕ, f and $f^\vee$ are unramified vectors,

$$Z(s + \frac{n-1}{2}, \phi, f, f^\vee) = L(s, \pi).$$

Given the local theorem, we make the following

Definition: For any π, set $L(s,\pi)$ to be

the GCD of the family $Z(s+\frac{n-1}{2},\phi,f,f^\vee)$.

By (iii), it gives the right answer for unramified reps. So it is not unreasonable.

Example: Assume π is supercuspidal. Then

$$Z(s,\phi,f,f^\vee) = \int_{Z\backslash G} \langle f^\vee, g\cdot f\rangle \cdot |\det(g)|^s \cdot$$

$$\left(\int_Z \phi(zg)\cdot |z|^{ns}\cdot \omega_\pi(z)\,dz\right)\,dg$$

This is entire, so that

$$L(s,\pi) = 1.$$

Local Functional Eqn

Fix additive character ψ of F. This gives an additive character on $M_n(F)$:

$$\psi \circ Tr : x \mapsto \psi(Tr(x)).$$

For an additive Haar measure dx on $M_n(F)$, have the Fourier transform relative to $\psi \circ Tr$:

$$\phi \mapsto \widehat{\phi}$$

Then one has:

$$\frac{Z(\frac{n+1}{2} - s, \widehat{\phi}, f^{\vee}, f)}{L(1-s, \pi^{\vee})} = \epsilon(s, \pi, \psi) \cdot \frac{Z(s + \frac{n-1}{2}, \widehat{\phi}, f, f^{\vee})}{L(s, \pi)}$$

for some local epsilon factor

$$\epsilon(s, \pi, \psi) = a \cdot q^{bs}.$$

When π is unramified, and $\psi \circ Tr$ has conductor $M_n(\mathcal{O})$, one has

$$\epsilon(s, \pi, \psi) = 1.$$

Summary:

By considering a family of zeta integrals which naturally extends the case treated by Tate, we have defined the local L-factor

$$L(s, \pi_v)$$

and the local epsilon factor

$$\epsilon(s, \pi, \psi),$$

together with a local functional eqn.

So given a cuspidal automorphic $\pi = \otimes_v \pi_v$, we could define

$$L(s, \pi) = \prod_v L(s, \pi_v)$$

$$\epsilon(s, \pi) = \epsilon(s, \pi_v, \psi_v)$$

"Cuspidal automorphicity" implies that the first product converges when $Re(s) \gg 0$. But is $L(s, \pi)$ nice?

Global Matrix Coefficients

Suppose that

$$\pi = \otimes_v' \pi_v \subset \mathcal{A}_0(G)$$

Then its contragredient $\pi^\vee$ is also cuspidal automorphic, so that

$$\pi^\vee \subset \mathcal{A}_0(G).$$

A $G(\mathbb{A})$-invariant pairing

$$\pi^\vee \otimes \pi \longrightarrow \mathbb{C}$$

can be given by the explicit integral

$$f^\vee \otimes f \mapsto \int_{Z(\mathbb{A})\backslash \mathsf{GL}_n(F)\backslash \mathsf{GL}_n(\mathbb{A})} f^\vee(h) \cdot f(h)\, dh.$$

Denote this linear form by $\langle -, - \rangle_{Pet}$.

Then the function (a global matrix coeff)

$$g \mapsto \langle f^\vee, g \cdot f \rangle_{Pet}$$

is explicitly given by

$$g \mapsto \int_{Z(\mathbb{A})\cdot \mathsf{GL}_n(F)\backslash \mathsf{GL}_n(\mathbb{A})} f^\vee(h) \cdot f(hg)\, dh.$$

Note that

$$\dim \mathsf{Hom}_{G(F_v)}(\pi_v^\vee \otimes \pi_v, \mathbb{C}) = 1,$$

for all v, and

$$\dim \mathsf{Hom}_{G(\mathbb{A})}(\pi^\vee \otimes \pi, \mathbb{C}) = 1$$

So, if we pick some

$$\langle -, - \rangle_v \in \mathsf{Hom}_{G(F_v)}(\pi_v^\vee \otimes \pi_v, \mathbb{C})$$

then we get a factorization

$$\langle f^\vee, f \rangle_{Pet} = \prod_v \langle f_v^\vee, f_v \rangle_v.$$

Global Zeta Integrals

Given cuspidal π and $\pi^\vee$, we can now define the global zeta integral

$$Z(s,\phi,f,f^\vee) = \int_{G(\mathbb{A})} \phi(g)\cdot\langle f^\vee, g\cdot f_v\rangle_{Pet}\cdot|\operatorname{det}(g)|^s \, dg$$

for $f \in \pi$, $f^\vee \in \pi^\vee$ and $\phi \in S(M_n(\mathbb{A}))$.

Because of the factorization

$$\langle f^\vee, f\rangle_{Pet} = \prod_v \langle f_v^\vee, f_v\rangle_v,$$

we obtain at least formally

$$Z(s,\phi,f,f^\vee) = \prod_v Z(s,\phi_v,f_v,f_v^\vee)$$

when ϕ, f and $f^\vee$ are factorizable.

This allows us to pass from global to local zeta integrals.

Main Global Theorem

(i) There is a c such that $Z(s, \phi, f, f^\vee)$ converges for all ϕ, f and $f^\vee$ whenever $Re(s) > c$.

(ii) $Z(s, \phi, f, f^\vee)$ has analytic continuation to $\mathbb{C}$.

(iii) There is a global functional eqn

$$Z(n - s, \widehat{\phi}, f^\vee, f) = Z(s, \phi, f, f^\vee).$$

Corollary: The global L-function

$$\Lambda(s, \pi) = \prod_v L(s, \pi_v)$$

has analytic continuation to $\mathbb{C}$ and satisfies a functional eqn

$$\epsilon(s, \pi) \cdot L(1 - s, \pi^\vee) = L(s, \pi).$$

Summary:

At this point, we have realized our goals for GL(n) by generalizing the paradigm of Tate's thesis. Thus we have produced many "nice" L-funcitons (to the extent that we know cuspidal representations exist!)

Is this the end of the story?

Summary:

At this point, we have realized our goals for GL(n) by generalizing the paradigm of Tate's thesis. Thus we have produced many "nice" L-funcitons (to the extent that we know cuspidal representations exist!)

Is this the end of the story?

Mmmm.......not quite..........

Let's examine the case $n = 2$, where the theory of L-functions of modular forms was developed by Hecke.

L-functions of Modular Forms

Given a cusp form of level 1 and even weight k,

$$f(z) = \sum_{n \geq 0} a_n(f) e^{2\pi i n z},$$

one may consider the Dirichlet series:

$$L(s, f) = \sum_{n \geq 1} \frac{a_n(f)}{n^s}.$$

The proof that this is nice relies on the fact that $L(s, f)$ is related to f by a Mellin transform:

$$\int_0^\infty f(iy) \cdot y^s \cdot \frac{dy}{y} = (2\pi)^{-s} \cdot \Gamma(s) \cdot L(s, f)$$

Proof: When $Re(s)$ is large, we have:

$$\Lambda(s, f) = \int_0^\infty f(iy) \cdot y^s \cdot \frac{dy}{y}.$$

But the RHS is convergent for all s (and thus gives (i)). This is because:

- $f(iy)$ is exponentially decreasing as $y \to \infty$, since f is cuspidal.

- since f is modular with respect to

  $$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

  we have:

  $$f(iy) = (-1)^{k/2} \cdot y^{-k} \cdot f(i/y).$$

  So as $y \to 0$, $f(iy) \to 0$ faster than any power of y.

To see (ii), note that

$$\begin{aligned}
&\Lambda(s, f) \\
=&\int_0^{\infty} f(iy) \cdot y^s \cdot \frac{dy}{y} \\
=&\int_0^{\infty} (-1)^{k/2} y^{-k} f(i/y)\, y^s \frac{dy}{y} \\
=&(-1)^{k/2} \cdot \int_0^{\infty} f(it)\, t^{k-s} \cdot \frac{dt}{t} \qquad (t = 1/y). \\
=&\Lambda(k-s, f)
\end{aligned}$$

What is the point?

By a well-known dictionary,

$$\{\text{cuspidal Hecke eigenforms } f\}$$

$$\updownarrow$$

$$\{\text{cuspidal representations } \pi \text{ of GL(2)}\}.$$

In this dictionary,

$$\Lambda(s, f) \leftrightarrow \Lambda(s, \pi).$$

But does Hecke's proof of the "niceness" of $\Lambda(s, f)$ translate to the proof by Godement-Jacquet of the "niceness" of $\Lambda(s, \pi)$ (for $n = 2$)?

If not, what does it translate to?

This suggests: an L-function may be amenable to the "zeta integral" treatment in more than one way!

Variants of $L(s, f)$

Let $f = \sum_n a_n q^n$ be as above.

Twist by characters

If χ is a Dirichlet character, then consider

$$L(s, f, \chi) = \sum_{n \geq 1} \frac{a_n \cdot \chi(n)}{n^s}.$$

If f is a Hecke eigenform, this L-function is "nice". This is proved via the integral representation:

$$L(s, f, \chi) \approx \int_0^\infty f(it) \cdot \chi(t) \cdot t^{s-1} d^\times t,$$

which is just a simple variant of Hecke's treatment of $L(s, f)$. Moreover,

$$L(s, f, \chi) = \prod_p \frac{1}{1 - a_p \cdot \chi(p) \cdot p^{-s}}.$$

Rankin-Selberg L-function

If

$$g = \sum_n b_n q^n,$$

is another cuspidal Hecke eigenform of level 1, then Rankin and Selberg independently considered

$$L(s, f \times g) = \sum_n \frac{a_n b_n}{n^s}.$$

They showed this is "nice" via the integral representation:

$$L(s, f \times g) \approx \int_{\mathcal{H}} f(z) \cdot g(z) \cdot E(z, s) \frac{dz}{Im(z)^2}$$

where $E(s, z)$ is a non-holomorphic Eisenstein series.

Moreover, if

$$L(s,f) = \prod_p \frac{1}{(1-a_{1,p}p^{-s})(1-a_{2,p}p^{-s})}$$

$$L(s,g) = \prod_p \frac{1}{(1-b_{1,p}p^{-s})(1-b_{2,p}p^{-s})},$$

then the local Euler factor at p of $L(s, f \times g)$ is

$$\prod_{i,j=1}^{2} \frac{1}{(1-a_{i,p}b_{j,p}p^{-s})}$$

These classical examples suggest:

At least for $n = 2$, one can obtain more nice L-functions from cuspidal π besides the standard L-function $L(s, \pi)$.

Indeed, it suggests that:

For cuspidal automorphic representations

$$\begin{cases} \pi_1 \text{ of } \mathrm{GL}(n_1) \\ \pi_2 \text{ of } \mathrm{GL}(n_2), \end{cases}$$

one might expect to define a "nice" L-function

$$L(s, \pi_1 \times \pi_2)$$

in some way.

Automorphic Rankin-Selberg L-function

Suppose

$$\pi_1 = \otimes_v \pi_{1,v} \quad \text{and} \quad \pi_2 = \otimes_v \pi_{2,v}.$$

Outside a finite set S of places of F:

$$\pi_1 \longrightarrow \{s_v \in \mathrm{GL}_{n_1}(\mathbb{C}) = \mathrm{GL}(V_1)\},$$

$$\pi_2 \longrightarrow \{t_v \in \mathrm{GL}_{n_2}(\mathbb{C}) = \mathrm{GL}(V_2)\}.$$

Now we have the tensor product rep of complex groups:

$$r : \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \longrightarrow \mathrm{GL}(V_1 \otimes V_2)$$

So we obtain:

$$\{r(s_v, t_v) = s_v \otimes t_v \in \mathrm{GL}(V_1 \otimes V_2)\}.$$

Definition:

$$L(s, \pi_{1,v} \times \pi_{2,v}) = \frac{1}{\det(1 - q_v^{-s} \cdot s_v \otimes t_v | V_1 \otimes V_2)}.$$

This allows one to define $L^S(s, \pi_1 \times \pi_2)$ $(Re(s) \gg 0)$.

Automorphic L-functions à la Langlands

The above construction can be generalized.

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic rep of $G(\mathbb{A})$. Then outside of a finite set S,

$$\pi \longrightarrow \{s_v \in \widehat{G}(\mathbb{C})\}.$$

Given a representation

$$r : \widehat{G}(\mathbb{C}) \longrightarrow \mathrm{GL}(V),$$

one obtains a collection of semisimple elements $r(s_v)$ for $v \notin S$, well-defined up to conjugacy.

Then one sets

$$L(s, \pi_v, r) := \frac{1}{\det(1 - q_v^{-s} r(s_v)|V)}$$

and

$$L^S(s, \pi, r) := \prod_{v \notin S} L(s, \pi_v, r).$$

Langlands' Conjecture

One can define $L(s, \pi_v, r)$ and $\epsilon(s, \pi, r)$ for general π_v, so that the global $\Lambda(s, \pi, r)$ is nice.

Examples: Let's take $G = \mathrm{GL}(n)$, so that $\widehat{G} = \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$. Some common r's are:

$$r = id : \mathrm{GL}(V) \longrightarrow \mathrm{GL}(V)$$

$$r = Sym^k : \mathrm{GL}(V) \longrightarrow \mathrm{GL}(\mathrm{Sym}^k V)$$

$$r = \wedge^k : \mathrm{GL}(V) \longrightarrow \mathrm{GL}(\wedge^k V).$$

Upshot: Nice L-functions are associated to pairs (π, r).

The Zeta Integral for $\mathbf{GL}(n) \times \mathbf{GL}(n-1)$

In the rest of the lecture, we will describe the relevant zeta integrals for the Rankin-Selberg L-functions. The analog of Tate's thesis was developed by Jacquet-Piatetski-Shapiro-Shalika.

Let π and π' be cuspidal reps of $\mathrm{GL}(n)$ and $\mathrm{GL}(n-1)$ resp. We consider

$$Z(s+1/2, f, f') =$$

$$\int_{\mathrm{GL}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(\mathbb{A})} f\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \cdot f'(h) \cdot |\det(h)|^s \, dh,$$

for $f \in \pi$ and $f' \in \pi'$.

When $n = 2$, observe that this is just the automorphic version of Hecke's classical work and was done by Jacquet-Langlands.

The Zeta Integral for $\mathbf{GL}(n) \times \mathbf{GL}(n)$

Suppose now π_1 and π_2 are two cuspidal reps of $\mathrm{GL}(n)$. The global zeta integral for this case is:

$$Z(s, f_1, f_2, \phi) =$$

$$= \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})} f_1(g) \cdot f_2(g) \cdot E(s, \phi)\, dg$$

where $E(s, \phi)$ is an Eisenstein series attached to a Schwarz function ϕ on $\mathbb{A}^n$.

Again, when $n = 2$, this is simply the automorphic analog of the classical work of Rankin-Selberg.

Convergence

Proposition: The zeta integrals above converges absolutely for all $s \in \mathbb{C}$ and thus define entire functions.

(Contrast this with Tate and Godement-Jacquet)

Consider the case GL(2) $\times$ GL(1). Let π be a cuspidal rep. of GL_2 and χ a Hecke character. Assume for simplicity that π has trivial central character.

The global zeta integral is:

$$Z(s+1/2, f, \chi) =$$

$$\int_{F^\times\backslash\mathbb{A}^\times} f\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \cdot \chi(t) \cdot |t|^s \, d^\times t.$$

Proof

Using

$$F^\times\backslash\mathbb{A}^\times \cong \mathbb{R}_+^\times \times (F^\times\backslash\mathbb{A}^1),$$

one has

$$\int_{F^\times\backslash\mathbb{A}^\times} = \int_0^\infty t^s \cdot \left(\int_{F^\times\backslash\mathbb{A}^1} f\begin{pmatrix} tx & 0 \\ 0 & 1 \end{pmatrix} dx\right) d^\times t.$$

The inner integral is over a compact set, and defines a function of t with the following properties:

(i) as $t \to \infty$, it decreases rapidly, since f is cuspidal;

(ii) as $t \to 0$, one has

$$\begin{aligned} & f\begin{pmatrix} tx & 0 \\ 0 & 1 \end{pmatrix} \\ =& f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} tx & 0 \\ 0 & 1 \end{pmatrix}\right) \\ =& f\left(\begin{pmatrix} 1 & 0 \\ 0 & tx \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\ =& (w \cdot f)\begin{pmatrix} 1 & 0 \\ 0 & tx \end{pmatrix} \\ =& (w \cdot f)\begin{pmatrix} 1/tx & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

which is rapidly decreasing as $t \to 0$.

Compare this with Hecke's classical argument.

Whittaker-Fourier coefficients

It is not clear why these zeta integrals factor into product of local integrals. It is also not clear what are the local zeta integrals.

Let f be an automorphic form on $G = GL_n$. If $N \subset G$ is a unipotent subgroup, say the unipotent radical of a parabolic subgroup, one can consider the Fourier coefficients of f along N.

Namely, if χ is a unitary character of $N(\mathbb{A})$ which is trivial on $N(F)$, we have

$$f_{N,\chi}(g) = \int_{N(F)\backslash N(\mathbb{A})} \overline{\chi(n)} \cdot f(ng)\, dn$$

Note that if N is abelian, then we have:

$$f(g) = \sum_{\chi} f_{N,\chi}(g).$$

We apply the above to the unipotent radical U of the Borel subgroup B of upper triangular matrices.

Definition: A character χ of $U(\mathbb{A})$ is **generic** if the stabilizer of χ in $T(\mathbb{A})$ is the center $Z(\mathbb{A})$ of $GL_n(\mathbb{A})$.

Examples:

(i) When $G = GL_2$, a generic character of $U(F)\backslash U(\mathbb{A})$ just means a non-trivial character of $F\backslash\mathbb{A}$. If we fix a character ψ of $F\backslash\mathbb{A}$, then all others are of the form

$$\chi_a(x) = \psi(ax)$$

for some $a \in F$.

(ii) When $G = GL_3$, a character of $U(\mathbb{A})$ trivial on $U(F)$ has the form

$$\chi_{a_1,a_2}\begin{pmatrix} 1 & x_1 & * \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} = \psi(a_1x_1 + a_2x_2)$$

for some a_1 and $a_2 \in F$.

Saying that χ_{a_1,a_2} is generic means that a_1 and a_2 are both non-zero.

Definition: A **Whittaker-Fourier coefficient** of f is a Fourier coefficient $f_{U,\chi}$ with χ generic.

Easy to see that the group $Z(F)\backslash T(F)$ acts simply transitively on the generic characters of $U(\mathbb{A})$ trivial on $U(F)$. If $t\cdot\chi = \chi'$ with $t \in T(F)$, then

$$f_{U,\chi'}(g) = f_{U,\chi}(t^{-1}g).$$

So $f_{U,\chi} \neq 0$ iff $f_{U,\chi'} \neq 0$ for generic χ and χ'.

Definition: A representation $\pi \subset \mathcal{A}(G)$ is said to be **globally generic** if there exists $f \in \pi$ whose Fourier-Whittaker coefficient $f_{U,\chi} \neq 0$ for some (and hence all) generic characters χ.

Equivalently, the linear form on π:

$$f \mapsto f_{U,\chi}(1)$$

is a nonzero element of

$$\mathrm{Hom}_{U(\mathbb{A})}(\pi, \mathbb{C}_\chi).$$

Whittaker functionals

One can define the notion of a "generic representation" locally.

Let π_v be a representation of $G(F_v)$ and let

$$\chi_v : U(F_v) \longrightarrow \mathbb{C}$$

be a generic unitary character.

Definition: π_v is an **abstractly generic** representation if

$$\mathrm{Hom}_{U(F_v)}(\pi_v, \mathbb{C}_{\chi_v}) \neq 0.$$

An element in this Hom space is called a local **Whittaker functional**.

Theorem (Local uniqueness of Whittaker functionals):

Let π_v be an irreducible smooth representation of $G(F_v)$. Then

$$\dim \mathrm{Hom}_{U(F_v)}(\pi_v, \mathbb{C}_{\chi_v}) \leq 1.$$

Fourier Expansion of a Cusp Form

Proposition: We have the expansion

$$f(g) = \sum_{\gamma \in U_{n-1}(F)\backslash GL_{n-1}(F)} f_{U,\chi}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Here U_{n-1} is the unipotent radical of the Borel subgroup of GL_{n-1} and χ is a generic character of U_n.

Corollary: A cuspidal rep of GL_n is globally generic.

Though not too deep, the proof of this proposition is quite intricate to execute, except when $n = 2$:

$$
\begin{aligned}
f(g) &= \sum_{\chi \neq 1} f_{U,\chi}(g) \\
&= \sum_{a \in F^\times} f_{U,\chi_a}(g) \\
&= \sum_{a \in F^\times} f_{U,\chi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)
\end{aligned}
$$

Euler Product of Zeta Integrals

Work with $\mathsf{GL}_n \times \mathsf{GL}_{n-1}$ case.

$$Z(s+1/2, f, f') =$$

$$= \int_{\mathsf{GL}_{n-1}(F)\backslash \mathsf{GL}_{n-1}(\mathbb{A})} \sum_{\gamma \in U_{n-1}(F)\backslash \mathsf{GL}_{n-1}(F)}$$

$$f_{U,\chi}\begin{pmatrix} \gamma h & 0 \\ 0 & 1 \end{pmatrix} \cdot f'(h) \cdot |\det(h)|^s \, dh.$$

$$= \int_{U_{n-1}(F)\backslash \mathsf{GL}_{n-1}(\mathbb{A})} f_{U,\chi}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \cdot f'(h) \cdot |\det(h)|^s \, dh$$

$$= \int_{U_{n-1}(\mathbb{A})\backslash \mathsf{GL}_{n-1}(\mathbb{A})} \int_{U_{n-1}(F)\backslash U_{n-1}(\mathbb{A})}$$

$$f_{U,\chi}(uh) \cdot f'(uh) \cdot |\,\mathsf{det}(h)|^s \, du \, dh$$

$$= \int_{U_{n-1}(\mathbb{A})\backslash \mathsf{GL}_{n-1}(\mathbb{A})} |\,\mathsf{det}(h)|^s \cdot f_{U,\chi}(h)$$

$$\cdot \left(\int_{U_{n-1}(F)\backslash U_{n-1}(\mathbb{A})} \chi(u) \cdot f'(uh) du \right) \, dh$$

$$= \int_{U_{n-1}(\mathbb{A})\backslash \mathsf{GL}_{n-1}(\mathbb{A})} |\,\mathsf{det}(h)|^s \cdot f_{U,\chi}(h) \cdot f'_{U',\chi'}(h) \, dh$$

with

$$\chi' = \chi^{-1}|_{U_{n-1}}$$

By the local uniqueness of Whittaker functionals,

$$f_{U,\chi}(h) = \prod_v W_v(h_v \cdot f_v)$$

for some

$$W_v \in \mathrm{Hom}_{N(F_v)}(\pi_v, \mathbb{C}_{\chi_v}).$$

This gives, at least formally,

$$Z(s, f, f') = \prod_v Z_v(s, f_v, f'_v)$$

where

$$Z_v(s + 1/2, f_v, f'_v) =$$

$$\int_{U(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} W_v(h \cdot f_v) \cdot W'_v(h \cdot f'_v) \cdot |\det(h)|^s \, dh.$$

It remains to develop the local theory for this family of local zeta integrals..........

Summary:

(i) We explained how (partial) automorphic L-functions $L(s, \pi, r)$ are defined, following Langlands.

(ii) We examined Rankin-Selberg L-functions for GL$(n) \times$ GL(m), following the paradigm of Tate's thesis.

(iii) We noted that a given L-function can be attacked by possibly more than one family of zeta integrals.

As for finding a zeta integral that actually works for a given $L(s, \pi, r)$, it is truly an art.