# First Joint Meeting between the RSME and the AMS

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# Abstracts

## Session 4

## Banach Spaces of Analytic Functions

Organizers:

Daniel Girela (Universidad de Málaga)
Michael Stessin (SUNY at Albany)

# Index of Abstracts

**(In this index, in case of multiple authors, only the speaker is shown)**

## *On $Q_p(R)$ for Riemann surfaces*

**Rauno Aulaskari*** (University of Joensuu)

**Huaihui Chen** (Nanjing Normal University)

The function classes $Q_p(R)$ on hyperbolic Riemann surfaces are considered. New methods for proving the nesting property, that is, $Q_p(R) \subset Q_q(R)$ for $0 < p < q$, are developed. Specially, the equality condition for the above inclusion is studied.

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## *Recent results related to the Bloch and Landau covering constants*

**Albert Baernstein II*** (University of Washington)

**Alexandre Eremenko** (Purdue University)

**Alexander Fryntov** (Purdue University)

**Alexander Yu. Solynin** (Russian Academy of Sciences, St. Petersburg)

We prove, among other examples, that if $f$ is holomorphic in the unit disk with $|f'(0)| = A|f(0)|$, where $A > 4$, then $f$ covers some annulus of the form $r < |w|$.

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## *On the modulus of mean oscillation*

**Óscar Blasco de la Cruz*** (Universidad de Valencia)

**M. Amparo Pérez** (Universidad de Valencia)

For $f \in L^1(\mathbb{T})$ and $t > 0$ we define

$$w_{mo}(f)(t) = \sup_{|I| \leq t} \frac{1}{|\ I\ |} \int_I |\ f(e^{i\theta}) - m_I(f)\ | \ \frac{d\theta}{2\pi}$$

and

$$w_{ho}(f)(t) = \sup_{1-t \leq |z| < 1} \int_{\mathbb{T}} |\ f(e^{i\theta}) - f(z)\ |\ P_z(e^{i\theta}) \frac{d\theta}{2\pi}$$

where $m_I(f) = \int_I f(e^{i\theta}) \frac{d\theta}{2\pi}$ and $f(z) = \int_{\mathbb{T}} f(e^{i\theta}) P_z(e^{i\theta}) \frac{d\theta}{2\pi}$.

Of course, $BMO(\mathbb{T})$ corresponds to the space of functions for which $w_{ho}(f)(t) < \infty$ or, equivalently, $w_{mo}(f)(t) < \infty$, for some (equivalently, for all) $t > 0$. For $1 \leq p < \infty$ we consider the spaces $OM^p(\mathbb{T})$ of functions satisfying $\int_0^1 w_{mo}(f)^p(t) \frac{dt}{t} < \infty$.

For $p_1 \leq p_2$ and $\alpha > 0$, one has

$$Lip_\alpha(\mathbb{T}) \subset OM^{p_1}(\mathbb{T}) \subset OM^{p_2}(\mathbb{T}) \subset VMO(\mathbb{T}).$$

Similarly we can define $OH^p(\mathbb{T})$ as the space of functions such that $\int_0^1 w_{ho}(f)^p(t) \frac{dt}{t} < \infty$.

The main result presented is that $OH^p(\mathbb{T}) = OM^p(\mathbb{T})$ and the norms are equivalent.

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## *Differential operators, the Laguerre-Pólya class and the Riemann $\xi$-function*

**George Csordas** (University of Hawaii)

The purposes of this lecture are (1) to formulate several related open problems involving infinite order differential operators and the distribution of zeros of certain transcendental entire functions, and (2) to present some recent results pertaining to these problems. The questions and results are concerned with the "movement" of the zeros of real entire functions when they are subjected to the influence of certain infinite order differential operators which depend on a parameter. The application of a special differential operator to the Riemann $\xi$-function gives rise to the *de Bruijn-Newman constant* constant $\Lambda$. It is known that the inequality $\Lambda \leq 0$ is equivalent to the Riemann Hypothesis. (Recently, G. Csordas, A.M. Odlyzko, W. Smith, and R. S. Varga have shown, using a spectacularly close pair of zeros of the Riemann zeta function (namely the 1,048,449,114th and 1,048,449,115th zeros), that $-5.895 \cdot 10^{-9} < \Lambda$.)

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## *Removable singularities for analytic functions in the little Zygmund class*

**Juan J. Donaire** (Universitat Autònoma de Barcelona)

The Zygmund space $\Lambda_*(\mathbb{C})$ is defined to be the set of bounded complex valued functions $f$ in $\mathbb{C}$ such that

$$\sup_{z,h\in\mathbb{C}} \frac{|f(z+h)+f(z-h)-2f(z)|}{|h|} < \infty\,.$$

An important subspace of $\Lambda_*$ is $\lambda_*(\mathbb{C})$, the little Zygmund class which consists on those Zygmund functions such that

$$\lim_{\delta\to 0} \sup_{z\in\mathbb{C},|h|\leq\delta} \frac{|f(z+h)+f(z-h)-2f(z)|}{|h|} = 0\,.$$

The problem in which we are interested is the description of those compact sets $K \subset \mathbb{C}$ that are removable for the analytic functions in $\lambda_*(\mathbb{C})$, that is if $f \in \lambda_*(\mathbb{C})$ and $f \in H(\mathbb{C}\setminus K)$ then $f \in H(\mathbb{C})$.

The description of removable singularities for another related spaces like $\mathrm{lip}_\alpha$ or VMO has been provided in terms of what is called *lower Hausdorff content*. O'Farrell proved that a compact set $K$ is $\mathrm{lip}_\alpha$-removable if, and only if, $M^*_{1+\alpha}(K) = 0$ and Verdera obtained the corresponding result for VMO by showing that $K$ is VMO-removable if, and only if, $M^*_1(K) = 0$.

Using techniques of martingales, we shall prove that if $\Phi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$, then the condition $H^*_\Phi(K) = 0$ is sufficient for the $\lambda_*$-removability of $K$. Moreover, we shall prove that the previous result is sharp in some sense.

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*Uniformly discrete sets and Bergman spaces*

**Peter Duren*** (University of Michigan)

**Alexander P. Schuster** (San Francisco State University)

**Dragan Vukotić** (Universidad Autónoma de Madrid)

A sequence $\{z_k\}$ of points in the unit disk $\mathbb{D}$ is said to be *uniformly discrete* if it is separated in the pseudohyperbolic metric: $\rho(z_j, z_k) \geq \delta > 0$ for all $j \neq k$. We study relations of uniformly discrete sets to zero-sets and interpolation sets for the Bergman spaces $A^p$. We also show that if $\mathbb{D}$ is the union of pseudohyperbolic disks $\Delta(z_k, \varepsilon)$ of sufficiently small pseudohyperbolic radius $\varepsilon$, then $\{z_k\}$ is a sampling set for $A^p$.

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*Holomorphic functions and quasiconformal mappings with smooth moduli*

**Konstantin M. Dyakonov** (Universitat Autònoma de Barcelona and Steklov Institute of Mathematics, St. Petersburg)

For a holomorphic function (resp., quasiconformal mapping) $f$ on a fairly general domain in $\mathbb{C}^n$ (resp., $\mathbb{R}^n$), we shall discuss the relationship between Lipschitz-type properties of $f$ and those of its modulus $|f|$.

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*Interpolation and approximation by rational functions with prescribed poles outside the unit circle*

**Adhemar Bultheel** (KU Leuven)

**Pablo González-Vera*** (Universidad de La Laguna)

**Erik Hendriksen** (University of Amsterdam)

**Olav Njåstad** (Norwegian University of Science and Technology)

¿From the Erdös-Turán Theorem, it can be deduced that if $f$ is a continuous function on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L_n(f, z)$ denotes the unique Laurent polynomial interpolating $f$ at the $(2n+1)$-th roots of unity, then

$$\lim_{n\to\infty} \int_{\mathbb{T}} |f(z) - L_n(f, z)|^2 |dz| = 0.$$

Several years later, Walsh and Sharma produced a similar result for a function analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\mathbb{D} \cup \mathbb{T}$ and making use of algebraic interpolating polynomials in the roots of unity.

In this talk, the above results will be extended in two directions. On the one hand, more general functions than polynomials or Laurent polynomials will be used and, on the other hand, the interpolation points are closely associated with certain orthogonal rational functions with respect to a given positive measure $\mu$ on $\mathbb{T}$. As an application, integrals of the form $\int_{\mathbb{T}} f(z) d\mu(z)$, $f$ being an analytic function on a region containing $\mathbb{T}$, can be efficiently estimated.

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## *Weak compactnes in $*$-invariant subspaces*

**John R. Akeroyd** (University of Arkansas)

**Dmitry Khavinson*** (University of Arkansas)

**Harold S. Shapiro** (Royal Institute of Technology, Stockholm)

The context of much of the work in this paper is that of a backward-shift invariant subspace of the form $K_B := H^2(\mathbb{D}) \ominus BH^2(\mathbb{D})$, where $B$ is some infinite Blaschke product. We address (but do not fully answer) the question: For which $B$ can one find a (convergent) sequence $\{f_n\}_{n=1}^{\infty}$ in $K_B$ such that the sequence of real measures $\{\log|f_n|d\theta\}_{n=1}^{\infty}$ converges weak-star to some nontrivial singular measure on $\partial\mathbb{D}$? We show that, in order for this to hold, $K_B$ must contain functions with nontrivial singular inner factors, and, in a rather special setting, we show that this is also sufficient. Much of the paper is devoted to finding conditions (on $B$) that guarantee that $K_B$ has no functions with nontrivial singular inner factors. Our main result in this direction is based on the "geometry" of the zero set of $B$.

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## *Interpolation in the Nevanlinna class and harmonic majorants*

**Andreas Hartmann** (Université de Bordeaux I)

**Xavier Massaneda** (Universitat de Barcelona)

**Artur Nicolau*** (Universitat Autònoma de Barcelona)

**Pascal Thomas** (Université Paul Sabatier)

We consider an interpolation problem in the Nevanlinna class and discuss its relation with the existence of certain harmonic majorants.

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## *Luecking's condition for zeros of Bloch function*

**Artur Kukuryka** (Maria Curie-Sklodowska University)

**Maria Nowak*** (Maria Curie-Sklodowska University)

In his paper *Zero sequences for Bergman spaces,* Complex Variables **30** (1996), 345–362, D. Luecking gave a characterization for zero-sets for Bergman space $A^p$, with $0 < p < \infty$, in terms of the subharmonic function $k$ defined for $z \in \mathbb{D}$ by

$$k(z) = \frac{|z|}{2} \sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^2}{|1-\bar{z}_n z|^2}.$$

He proved that $\{z_n\}$ is an $A^p$ zero-set if, and only if, there is a harmonic function $h$ such that $e^{pk+h} \in L^1(\mathbb{D})$. We show that a necessary condition for zero-sets of Bloch functions, similar to that above, can be obtained and we prove the following theorem.

If $\{z_n\}$ is a sequence of zeros of a Bloch function and $k$ is given as above, then there is a function $h$ harmonic on $\mathbb{D}$ and such that

$$e^{k(z)+h(z)} = O\left(\log\frac{1}{1-|z|}\right) \quad \text{as} \quad |z| \to 1.$$

We also show how known necessary conditions for zero-sets for Bergman spaces and Bloch space can be derived from the above results.

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## *A unified approach to Farrell and Mergelyan sets*

**Fernando Pérez-González*** (Universidad de La Laguna)

**Daniel Suárez** (Universitat Autònoma de Barcelona)

We extend the notions of Farrell and Mergelyan set for spaces of analytic functions and unify many results of the literature. We also prove that Farrell sets are Mergelyan sets under very general conditions

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## *Schwarz-Christoffel mapping of multiply connected domains*

**John Pfaltzgraff** (University of North Carolina)

A Schwarz-Christoffel mapping formula is established for polygonal domains of finite connectivity $m \geq 2$ thereby extending the results of Christoffel (1867) and Schwarz (1869) for $m = 1$ and Komatu (1945), for $m = 2$. A formula for $f$, the conformal map of the exterior of $m$ bounded disks to the exterior of $m$ bounded, disjoint polygons, is derived. The derivation characterizes the global pre-Schwarzian $f''(z)/f'(z)$ on the Riemann sphere in terms of its singularities on the sphere and its values on the $m$ boundary circles via the reflection principle, and then identifies a singularity function with the same boundary behavior. The singularity function is constructed by a "method of images" type of infinite sequence of iterations of reflecting prevertex singularities from the $m$ boundary circles to the whole sphere.

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## *Harmonic measure on Swiss cheeses with uniformly spaced holes*

**Joaquim Ortega-Cerdà** (Universitat de Barcelona)

**Kristian Seip*** (Norwegian University of Science and Technology)

We consider domains obtained by removing from the unit disk little disks around each point of a separated sequence. The sequence is assumed to be uniformly dense, which means that there is a uniform bound on the distance from any point to the sequence. A necessary and sufficient condition for the outer boundary to have positive harmonic measure at any point of such a domain is given. This condition coincides with a condition of Lyubarskii and Seip describing so-called essential minorants for $H^\infty$. It is also shown how lower and upper uniform densities can be expressed in terms of similar harmonic measures; such densities describe sampling and interpolating sequences for Bergman spaces.

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*Hausdorff matrices acting on spaces of analytic functions*

**Aristomenis G. Siskakis** (University of Thessaloniki)

We consider Hausdorff matrices of the form $H = H_\mu = (c_{n,k})$ where

$$c_{n,k} = \begin{cases} \binom{n}{k}\Delta^{n-k}\mu_k, & k \leq n, \\ 0 & k > n, \end{cases}$$

$\Delta$ is the forward difference operator $\Delta\mu_n = \mu_n - \mu_{n+1}$ on scalar sequences, and $\mu_n$ a moment sequence

$$\mu_n = \int_0^1 t^n \, d\mu(t), \qquad n = 0, 1, \cdots,$$

for a positive finite Borel measure $\mu$ on $(0, 1]$. These matrices are then allowed to act on the sequence of Taylor coefficients of functions analytic on the disc. We find conditions on $\mu$ so that they represent bounded operators on spaces of analytic functions such as Hardy spaces.

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*Equivalence of summatory conditions along sequences for bounded holomorphic functions*

**Vladimir Ya. Eiderman** (Moscow State Civil Engineering University)

**Pascal Thomas*** (Université Paul Sabatier)

A sequence $\{z_k\} \subset \mathbb{D}$ is called *$\theta$-thin*, if there is a nonzero $f \in H^\infty(\mathbb{D})$ with

$$\sum_k \rho_\theta(1 - |z_k|)|f(z_k)| < \infty,$$

where $\rho_\theta(t) := te^{\theta(t)}$. A non-$\theta$-thin sequence is called *$\theta$-thick*. All sequences will be assumed hyperbolically separated. Let $\theta_i \geq 0$ be continuous and nonincreasing for $i = 1, 2$.

a) If $\theta_1(t) \approx \theta_2(t)$, then $\theta_1$ and $\theta_2$-thin sequences coincide.

b) If $\theta_1(t)/\theta_2(t) \to \infty$ as $t \to 0$ and $\int_0 \frac{dt}{t\mathbf{t}_2(t)} = \infty$ (so there are non-Blaschke $\theta_2$-thin sequences), then there exists a sequence $\{z_k\}$ which is $\theta_2$-thin and $\theta_1$-thick.

Consider now functions $\theta_1$ and $\theta_2$ bounded from above and write $\rho_i = \rho_{\theta_i}$ for $i = 1, 2$. If $\rho_1(t)$ is continuous, nondecreasing, and verifies $\rho_1(t) \leq Ct$,

$$\int_0 \frac{\rho_1(t)\, dt}{t2} = \infty,$$

(so there are $\theta_1$-thick sequences) and $\rho_1(t)/\rho_2(t) \to \infty$ as $t \to 0$, then there is a $\theta_1$-thick sequence $\{z_k\}$ such that the inequality defining $\theta$-thinnes above holds with $\rho = \rho_2$ with all $f \in H^\infty$.

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*Some thoughts on Toeplitz operators on the Hardy space*

**Dragan Vukotić** (Universidad Autónoma de Madrid)

Since the well known work of Brown and Halmos from the 60's, the products and commutators of Toeplitz operators have been studied extensively by a number of authors, both on the Hardy and on the Bergman space. In spite of several recent advances, various interesting problems in this field remain open.

In this preliminary report we show how some simple properties related to the commutators of two Toeplitz operators on the Hardy space can be used in the study of products of such operators. We also review the current state of some open problems.

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*Finite type subnormals, vector Toeplitz and separating real algebraic curves*

**Dmitry V. Yakubovich** (Universidad Autónoma de Madrid)

An operator $S$ on a Hilbert space $H$ is called subnormal if there is a larger Hilbert space $K$ such that $S$ is a restriction to $H$ of a normal operator which acts on $K$. A real algebraic curve $\Delta$ in $\mathbb{C}^2$ will be called separating if the set of its real points separates, in the topological sense, each of its irreducible components. In the works by Xia, McCarthy, Yang and the speaker, an intimate connection between subnormal operators whose self-commutator has finite rank and separating algebraic curves was established. It leads to a functional model of $S$ as the multiplication operator by one of the complex coordinates in $\mathbb{C}^2$ on $H^2$ of a "half" of $\Delta$. In this talk, we will give a connection between these objects and a class of classical Toeplitz operators with rational matrix symbol. This connection gives rise to a new method for parametrizing real separating algebraic curves, which also might be of interest to specialists in algebraic geometry

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*Complemented Invariant Subspaces in Bergman Spaces*

**Kehe Zhu** (University SUNY at Albany)

For $0 < p < \infty$ let $A^p$ denote the Bergman $L^p$ space in the open unit disk of the complex plane. A closed subspace $X$ of $A^p$ is said to be invariant if it is invariant under multiplication by the coordinate function $z$. The question under consideration is this: which invariant subspaces of $A^p$ are complemented in $A^p$? Two classes of invariant subspaces will be discussed in the talk, namely, zero-based invariant subspaces and invariant subspaces generated by inner functions with a single point mass on the boundary.