ON THE EQUATION $\mathsf{V} ⓜ \mathsf{G} = \mathsf{EV}$

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Abstract. This talk is a contribution to understanding the solutions V of the equation of pseudovarieties of semigroups $\mathsf{V} ⓜ \mathsf{G} = \mathsf{EV}$. Pseudoidentities are obtained guaranteeing that all pseudovarieties satisfying them are solutions or all (not too small) are counterexamples. Even within the pseudovariety J of all finite $\mathcal{J}$-trivial semigroups, there is a continuum of subpseudovarieties that fail the equation, thus invalidating a statement of Higgins and Margolis. A complete list of critical counterexample generators is given in the sense that a pseudovariety contains a subpseudovariety failing the equation if and only if it contains at least one of the semigroups in the list. Moreover, a pseudovariety excludes all semigroups in the list if and only if it consists of semigroups in which every product of idempotents is idempotent.

1. Introduction

Emerging mainly from work of Krohn and Rhodes [8, 9], on the so-called group complexity, and Schützenberger [12], on the syntactical characterization of special classes of formal languages, a general framework for the connections between finite semigroups, finite automata and rational languages was proposed by Eilenberg [5], which in turn prompted significant interest and led to many further developments (cf. [10, 2, 11]). According to Eilenberg's theory, finite semigroups should be viewed as being arranged in pseudovarieties, that is, classes closed under taking homomorphic images, subsemigroups, and finite direct products.

A key property in the applications that a pseudovariety may or may not enjoy is decidability, which means that the membership problem is algorithmically solvable. Several algebraic constructions yield natural operators on pseudovarieties, among which the semidirect ($*$) and Mal'cev (ⓜ) products are particularly important. The fact that neither of them preserves decidability [1, 4] makes the theory both complicated and rich as instances of decidability require adequately tailored methods.

There are also some natural operators that preserve decidability, such as the operator E associating with a pseudovariety V the pseudovariety of all finite semigroups whose idempotents generate a subsemigroup that belongs to V. Finding instances in which the two types of operators turn out to coincide may give particularly simple algorithmic solutions to specific pseudovariety membership problems. Denoting by G the pseudovariety of all finite

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groups, the two equations

$$\mathsf{V} * \mathsf{G} = \mathsf{EV} \tag{1}$$

$$\mathsf{V} \,ⓜ\, \mathsf{G} = \mathsf{EV} \tag{2}$$

express such connections between operators. Neither of them holds universally and (1) implies (2) since the inclusions $\mathsf{V} * \mathsf{G} \subseteq \mathsf{V} ⓜ \mathsf{G} \subseteq \mathsf{EV}$ always hold. Important examples of positive examples for both equations are the pseudovarieties Sl of all finite semilattices and J of all finite $\mathcal{J}$-trivial semigroups. Equation (1) has also been investigated for instance in [3] while information on the significance of the equation $\mathsf{V} * \mathsf{G} = \mathsf{V} ⓜ \mathsf{G}$ may be found in [6].

The main focus of concern in this talk is the equation (2), that is, finding pseudovarieties V that are solutions or that are counterexamples. It was prompted by a side remark in a paper by Higgins in Margolis [7] stating "it is known" that every pseudovariety V in the interval $[\mathsf{Sl}, \mathsf{DA}]$ satisfies (2). Here, DA is the pseudovariety of all finite semigroups in which the idempotents are the only regular elements. The Higgins-Margolis statement is not supported in any justification or bibliography and we show that in fact there is an uncountable subinterval of counterexamples.

2. Results

Consider the following semigroups:

- the semigroup with zero presented by $A_0 = \langle e, f \mid e^2 = e, f^2 = f, ef = 0\rangle$;
- for each prime p, the Rees matrix semigroup $K_p = \mathcal{M}\big(\mathbb{Z}/p\mathbb{Z}; 2, 2, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$.

For a pseudoidentity $u = v$, $[\![u = v]\!]$ is the pseudovariety consisting of all finite semigroups satisfying $u = v$. Let $\mathsf{B} = [\![x^2 = x]\!]$ be the pseudovariety of all finite bands.

Theorem 1. *The following are equivalent for a pseudovariety* V*:*

(i) V *contains a pseudovariety that fails the equation (2);*
(ii) V *contains at least one of the semigroups* A_0 *or* K_p *for a prime* p*;*
(iii) V *is not contained in* EB*.*

In fact, for every monoidal subpseudovariety (that is, one generated by its monoids) V of EB not contained in G, the equation (1) also holds.

For the next results, another couple of pseudovariety operators are needed. For a pseudovariety V, LV consists of all finite semigroups S such that the monoid eSe belongs to V for every idempotent $e \in S$. If H is a pseudovariety of groups, then $\overline{\mathsf{H}}$ consists of all finite semigroups all of whose subgroups belong to H.

An identity $u = v$ is balanced if v is obtained from the word u by permutation of the letter positions. For a finite semigroup S, $\mathsf{V}(S)$ is the smallest pseudovariety containing S.

Theorem 2. *Let* $u = v$ *be either a balanced identity or the pseudoidentity* $x^n = x^{\omega+n}$ *for some* n*. Then every pseudovariety* V *in the interval* $\big[\mathsf{V}(A_0), \mathsf{L}[\![u = v]\!]\big]$ *fails the equation (2).*

Note that $A_0 \in \mathsf{L}[\![u = v]\!]$ whenever $u = v$ is a pseudoidentity in which both sides use the same letters. So, there are plenty of intervals satisfying the conditions of Theorem 2.

For a prime q, $\mathsf{G}_{q'}$ denotes the pseudovariety of all finite q'-groups, that is, finite groups with no elements of order q.

Theorem 3. *Let p and q be primes such that p divides $q-1$. Then every pseudovariety V in the interval $[\mathsf{V}(K_p), \overline{\mathsf{G}_{q'}}]$ fails the equation (2).*

Note that, by Dirichlet's prime number theorem, for every prime p there are infinitely many primes q such that p divides $q-1$. Moreover, for such a pair of primes, we have $K_p \in \overline{G_{q'}}$. Thus, there are plenty of intervals satisfying the conditions of Theorem 3.

Let $\mathsf{N} = [\![x^\omega = 0]\!]$ be the pseudovariety of all finite nilpotent semigroups.

Theorem 4. *All the intervals in Theorems 2 and 3 have cardinality $2^{\aleph_0}$. More precisely, for each of the intervals $[\mathsf{U}, \mathsf{W}]$ in question, $\mathsf{N} \subseteq \mathsf{W}$ and there is a continuum of pseudovarieties of the form $\mathsf{U} \vee \mathsf{V}$ with $\mathsf{V} \subseteq \mathsf{N}$.*

In particular, the interval $\big[\mathsf{V}(A_0), \mathsf{DA} \cap \mathsf{LB}\big]$ has cardinality $2^{\aleph_0}$ and consists of counterexamples to the Higgins-Margolis statement.

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