GRÖBNER BASES IN CATEGORICAL ALGEBRA AND DIFFERENTIAL GEOMETRY

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Abstract. We will highlight how Gröbner bases played a key role to solve some interesting problems in categorical algebra and differential geometry.

1. Lie algebras and categorical algebra

The variety of Lie algebras is usually one of the central examples in the non-associative algebras world. Many properties, studied from a universal algebra point of view, were firstly introduced to Lie algebras and then generalised to several different structures. One of the many aims of categorical algebra is to give general definitions that can help to understand these notions and their generalisations.

We found that there are two interesting properties, *representability of actions* and the existence of *algebraic exponents*, that actually characterise Lie algebras amongst all varieties of non-associative algebras. To do so, we made use of computational algebra, Gröbner bases in particular.

2. Einstein manifolds

A pseudo-Riemanninan manifold (M, g) is said to be *Einstein* if its Ricci tensor is a multiple of the metric. These kind of metrics are optimal, in the sense that its scalar curvature is uniformally distributed through the manifold. In dimension 2 all pseudo-Riemannian manifold are Einstein, whereas in dimension 3 it is equivalent to have a constant sectional curvature and therefore they are completely determined $(\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3)$. However, in dimension 4 the classification of Einstein manifolds is an open problem.

We were able to classify an interesting subclass of Einstein manifolds in dimension 4, the homogeneous conformally Einstein ones. In fact, we showed that there is only one homothetic class of homogeneous strictly Bach-flat manifolds. To achieve this, we used Gröbner bases as a tool to identify and explore the polynomial ring formed by the parameters that arise from the defining Lie groups.

References

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