# FACTORIZATION OF THIRD ORDER ALGEBRO-GEOMETRIC ODOS ON SPECTRAL CURVES

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Abstract. We consider the classical factorization problem of a third order ordinary differential operator $L-\lambda$, for a spectral parameter $\lambda$. It is assumed that $L$ is an algebro-geometric operator, i. e. it has a nontrivial centralizer, which can be seen as the affine ring of an algebraic curve, the famous *spectral curve* $\Gamma$. We give a symbolic algorithm, using differential subresultants, to factor $L-\lambda_0$ for all but a finite number of points $P=(\lambda_0,\mu_0,\gamma_0)$ of $\Gamma$ based on the ring structure of the centralizer.

## Introduction

The factorization of ordinary differential operators, from the point of view of symbolic computation, has attracted much attention at least for a couple of decades, see for instance [1, 8, 15, 14], just to name a few. For the factorization of second order algebro-geometric ordinary differential operator $L$, a new approach was recently presented in [10] to factorize $L-\lambda$. It is indeed the centralizer, the set of all operators commuting with a given operator $L$, the structure that guaranties an effective factorization of $L-\lambda$, for an spectral parameter $\lambda$.

Continuing with this line of work, in this occasion we consider the effective factorization problem of $L-\lambda$ for an ordinary third-order differential operator

$$L=\partial^3+u_1\partial+u_0, \tag{1}$$

with (stationary) potentials $u_0$, $u_1$ in a differential field $K$, with derivation $\partial$ and field of constants $\mathbb{C}$, the field of complex numbers. The potentials $u_0$ and $u_1$ are assumed to be solutions of a stationary Boussinesq system [6].

Boussinesq systems have been widely studied, especially their rational solutions [4, 16]. They generate a hierarchy of integrable equations, the *Boussinesq hierarchy*, one of the Gelfand and Dickii integrable hierarchies of equations associated to differential operators of any order [5]. The stationary version of the Boussinesq hierarchy ultimately gives families of differential polynomials, in the coefficients of $L$, that are conditions for the existence of a nontrivial operator $A$ commuting with $L$.

The Burchnall and Chaundy Theorem [2] establishes a correspondence between pairs of commuting differential operators, $L$ and $A$, and algebraic curves, their spectral curve $\Gamma$,

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classically defined by the so called Burchnall and Chaundy (BC) polynomial. It is another famous result, Schur's Theorem [13], the one ensuring that centralizers have quotient fields that are function fields of one variable, therefore they can be seen as affine rings of curves, and in a formal sense these centralizers are *spectral curves*.

This is the approach followed in our work. We develop a factorization algorithm for $L-\lambda_0$, $\lambda_0\in\mathbb{C}$, with $L\in K[\partial]$ as in (1), for almost every point $P_0=(\lambda_0,\mu_0,\gamma_0)$ of the spectral curve $\Gamma$ of $L$. For this purpose, we have to establish an appropriate theoretical framework by means of Goodearl's results on centralizers, [7].

## 1. Factorization on the spectral curve

Let $K$ be a differential field with constants field algebraically closed of zero characteristic. We present the third order operators associated to classical Boussinesq systems as treated in [6]. In consequence, we rewrite $L$ as

$$L_3=\partial^3+q_1\partial+\frac{1}{2}q_1'+q_0. \tag{2}$$

Using the notation of [6], we consider a differential recursion given by two sequences of differential polynomials $f_{n,i},g_{n,i}$ in the ring of differential polynomials $\mathbf{C}\{u_0,u_1\}$. By direct computation we verify that:

$$Bsq_{3n+3+i}=\mathcal{R}Bsq_{3n+i},\ \text{ with } Bsq_{3n+i}=\begin{pmatrix}3\partial f_{n,i}\\ 3\partial g_{n,i}\end{pmatrix}\ \text{ for } i=1,2, \tag{3}$$

and initial conditions $(f_{0,1},g_{0,1})=(0,1)$, $(f_{0,2},g_{0,2})=(1,0)$, and we define the vectors:

$$v_{n+1,i}=\mathcal{R}^*v_{n,i},\ \ v_{n,i}=\begin{pmatrix}f_{n,i}\\ g_{n,i}\end{pmatrix}\ \text{ and } v_{0,1}:=\begin{pmatrix}0\\1\end{pmatrix},v_{0,2}:=\begin{pmatrix}1\\0\end{pmatrix}, \tag{4}$$

for matrices of pseudifferential operators:

$$\mathcal{R}=\begin{pmatrix}\mathcal{R}_1 & \mathcal{R}_2\\ \mathcal{R}_3 & \mathcal{R}_4\end{pmatrix}\ ,\ \mathcal{R}^*=\partial^{-1}\mathcal{R}\partial=\begin{pmatrix}\partial^{-1}\mathcal{R}_1\partial & \partial^{-1}\mathcal{R}_2\partial\\ \partial^{-1}\mathcal{R}_3\partial & \partial^{-1}\mathcal{R}_4\partial\end{pmatrix} \tag{5}$$

and pseudodifferential operators $\mathcal{R}_1=3q_0+2q_0'\partial^{-1}$, $\mathcal{R}_2=2\partial^2+2q_1+q_1'\partial^{-1}$, $\mathcal{R}_4=3q_0+q_0'\partial^{-1}$, and $\mathcal{R}_3=-\frac{1}{6}\partial^4-\frac{5}{6}q_1\partial^2-\frac{5}{4}q_1'\partial-\frac{2}{3}q_1^2-\frac{3}{4}q_1''+\left(-\frac{2}{3}q_1q_1'-\frac{1}{6}q_1'''\right)\partial^{-1}$.

Whenever the coefficients of L, $(u_0,u_1)=(q_1,\frac{1}{2}q_1'+q_0)\in K\times K$, satisfy a Boussinesq system $Bsq_\ell=\mathbf{0}$, we say that L is *a Boussinesq operator*. From now on we will consider $L$ to be a Boussinesq operator.

Next, let $A$ be a differential operator in the centralizer of $L$ in $K[\partial]$. We are interested in the common solutions of the system of linear differential equations

$$(L-\lambda)\psi=0,(A-\mu)\psi=0.$$

The tools we have chosen to study this problem are the differential resultant and the differential subresultants due to the following theorem of E. Previato.

**Theorem 1.1** (E. Previato, [11]). *Given $P,Q\in K[\partial]$ such that $[P,Q]=0$ then*

$$g(\lambda,\mu)=\partial\mathrm{Res}(P-\lambda,Q-\mu)\in C[\lambda,\mu]$$

*and also $g(P,Q)=0$. Hence $g$ is a defining polynomial for the spectral curve associated to the pair $P$, $Q$.*

Then we get the factorization formula (6). Moreover, the function $\phi$ in the next theorem can be effectively computed by means of differential subresultants [3]. See [12] for more details.

**Theorem 1.2.** *Let $L$ be a geometrically reducible Boussinesq operator, and $\Gamma$ its spectral curve. There exists a rational function $\phi \in K(\Gamma)$ such that, for every point $P_0 = (\lambda_0, \mu_0, \gamma_0)$ in $\Gamma \backslash Z$, the operator $L - \lambda_0$ has as right factor $\partial + \phi(P_0) = \gcd(\mathrm{L} - \lambda_0, \mathrm{A}_1 - \mu_0, \mathrm{A}_2 - \gamma_0)$.*

*Moreover, the following formula can be easily verified in $K[\partial]$:*

$$L - \lambda_0 = (\partial^2 + \phi(P_0)\partial + \phi(P_0)^2 + 2\phi(P_0)' + u_1)(\partial + \phi(P_0)). \tag{6}$$

To illustrate our results, we perform our methods on $L = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + h$, with $h \neq 0$, to obtain $A_1 = \partial^4 - \frac{8}{x^2}\partial^2 + \frac{24}{x^3}\partial - \frac{24}{x^4}$ and $A_2 = \partial^5 - \frac{10}{x^2}\partial^3 + \frac{40}{x^3}\partial^2 - \frac{80}{x^4}\partial + \frac{80}{x^5}$ in the centralizer of $L$. An effective computation of the basis of the centralizer of $L$ is the first step towards an effective Picard-Vessiot theory for spectral problems iniciated in [10] and [9] for second order operators. We will address this topic in a near future work.

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