THE DIAMETER OF THE COMMUTATION GRAPH OF A PERMUTATION

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Abstract. Using the standard Coxeter presentation for the symmetric group Sn, two reduced expressions for the same group element w are said to be commutationally equivalent if one expression can be obtained from the other one by applying a finite sequence of commutations. The commutation classes can be seen as the vertices of a graph C(w), where two classes are connected by an edge if elements of those classes differ by a long braid relation. We establish a statistic on the classes of C(w), inducing a rank poset structure on C(w) with a unique minimal and a unique maximal element. This allows us to give a precise formula for the diameter of the graph C(w). We recover, as special cases, the diameter of the commutation graph for the longest element of the symmetric group and the characterization of fully commutative permutations obtained by S. Billey, W. Jockusch and R. Stanley.

Introduction

Given an integer $n \geq 2$, we let S_n denote the symmetric group on the alphabet $[n] = \{1, \ldots, n\}$, with composition of permutations performed from right to left. The symmetric group S_n is an example of the more general concept of a Coxeter group, which are groups G that can be generated by a set $S = \{s_1, \ldots, s_m\} \subset G$ satisfying relations $(s_i s_j)^{m_{ij}} = 1$, where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. Any element $w \in G$ can be written as a finite product of elements of S. If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with ℓ minimal, the word $i_1 i_2 \cdots i_\ell$ is called a *reduced word* (or *reduced decomposition*) of g. In this case, we define the *length* of w by $\ell(w) = \ell$. The set of all reduced words of w is denoted by $\mathrm{R}(w)$.

The symmetric group has a Coxeter representation with generators s_i, the adjacent transposition interchanging the elements i and $i+1$, for $1 \leq i \leq n-1$, which satisfy the Coxeter relations

$$s_i s_j = s_j s_i \text{ for } |i-j| \geq 2, \tag{1}$$

$$s_i s_{i+1} s_1 = s_{i+1} s_i s_{i+1} \text{ for } 1 \leq i \leq n-2, \tag{2}$$

and $s_i^2 = 1$, the identity element. The relations (1) are known as *commutations* or *short braid relations*, and the relations (2) are called *long braid relations*.

The graph $G(w)$, having vertex set $\mathrm{R}(w)$ and a edge connecting two reduced words if they differ by a single Coxeter relation has been considered by several authors. Contracting the commutation edges of $G(w)$ leads to the associated graph $C(w)$, known as the commutation graph of w, which has also received some attention. Elnitsky [2] established a bijection

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between the vertices of $C(w)$ and rhombic tilings of certain polygons, and proved that $C(w)$ is bipartite. The diameter of $C(w_o)$ has been computed in [3], and a connection to geometric representation theory has been explored.

In this talk, we establish a statistic on the classes of $C(w)$, inducing a rank poset structure on $C(w)$ with a unique minimal and a unique maximal element. This allows us to give a precise formula for the diameter of the graph $C(w)$. We recover, as special cases, the diameter of the commutation graph for the longest element w_o and the characterization of fully commutative permutations obtained by Billey, Jockusch and Stanley [1].

1. A statistic on $C(w)$

The Rothe diagram for $w = w_1w_2...w_n$, denoted by $\mathbb{D}(w)$, gives a graphical representation of the inversion pairs of w and can be obtained by writing w vertically along the y-axis, with w_i at height i, and writing the positive numbers along the x-axis. Then, with this numerical arrangement of rows and columns, place a cell in position (w_j, i) whenever this is an inversion pair for w, for each $i, j \in n$. The cells of $\mathbb{D}(w)$ correspond to inversions in w, namely $(p, q) \in \mathbb{D}(w)$ if and only if (w_p, q) is an inversion of w.

A labelling of the cells of the Rothe diagram for a permutation $w \in S_n$ with the positive integers in $[\ell(w)]$ is called a *standard balanced tableaux* if for any entry of the diagram, the number of entries to its right that are greater is equal to the number of entries above it that are smaller. Denote the set of all standard balanced tableaux on $\mathbb{D}(w)$ by $\mathrm{SBT}(w)$.

S. Fomin *et al.* [5] stablished a bijection $a \mapsto P_a$ between $\mathrm{R}(w)$ and $\mathrm{SBT}(w)$, and we can define maps $\mathfrak{c}_i$ and $\mathfrak{b}_i$ in $\mathrm{SBT}(w)$ that translates to the tableaux setting the short and long relations defined in $\mathrm{R}(w)$.

Given a permutation $w \in {}_n$, define the set of *inversion triples*

$$\mathrm{T}_w = \{(w_k, w_j, w_i) : w_i > w_j > w_k \text{ and } i < j < k\},$$

form by all triples (x, y, z) in $[n]$ such that (z, y), (z, x) and (y, x) are inversions for w.

Definition 1.1. Given a permutation $w \in {}_n$, define the map Γ on the cartesian product $Red(w) \times T_w$ by setting

$$\Gamma(a, (x, y, z)) = \begin{cases} 1, & \text{if } P_a(y, x) > P_a(z, y) \\ 0, & \text{if } P_a(y, x) < P_a(z, y) \end{cases},$$

where $P_a(y, x)$ denotes the label in position (y, x) of P_a.

The Γ function is class invariant, that is two reduced words $a, b \in Red(w)$ are in the same commutation class if and only if $\Gamma(a, (x, y, z)) = \Gamma(b, (x, y, z))$, for all triple $(x, y, z) \in \mathrm{T}_w$. Moreover, there is one and only one commutation class $[a_{min}]$ (resp. $[a_{max}]$) in $C(w)$ for which $\Gamma(a_{min}, (x, y, z)) = 0$ (resp. $\Gamma(a_{min}, (x, y, z)) = 1$) for all $(x, y, z) \in T_w$.

Definition 1.2. Given $a, b \in Red(w)$, let

$$t(a, b) = \sum_{(x,y,z)\in \mathrm{T}_w} \Gamma(a, (x, y, z)) \oplus_2 \Gamma(b, (x, y, z)),$$

where $\oplus_2$ represents the sum modulo 2.

The map $t : R(w) \to \{1, 2, \ldots, |T_w|\}$ defined by $t(a) = t(a_{min}, a)$ is a rank function for the graph $C(w)$.

Proposition 1.3. *Let $w \in S_n$. The partial order defined on the commutation classes of $C(w)$ given by the transitive closure of covering relations*

$$[a] < [b] \text{ if } [a] \underset{\mathsf{L}}{\sim} [b] \text{ and } t(b) = t(a) + 1,$$

makes $C(w)$ into a ranked partially ordered set with a unique minimal element $[a_{\min}]$ and a unique maximal element $[a_{\max}]$.

Using this rank function, we can give a formula for the diameter of $C(w)$.

Theorem 1.4. *The diameter of $C(w)$ is equal to the cardinality of T_w.*

We recover, as special cases, the diameter of the commutation graph for the longest element of the symmetric group and the characterization of fully commutative permutations obtained by S. Billey, W. Jockusch and R. Stanley.

Corollary 1.5 (Billey, Jockusch, and Stanley, 1993)**.** *A permutation $w \in S_n$ is fully commutative if and only if it is 321-avoiding.*

Corollary 1.6. *The diameter of the commutation graph for the longest permutation w_0 of S_n is $\binom{n}{3}$.*

References

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