CALCULUS OF VARIATION PROBLEMS FOR VARIABLE ORDER GENERAL FRACTIONAL CALCULUS

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ABSTRACT. In this talk we combine two ideas: fractional derivatives of variable order and fractional derivatives depending on an another function. With such operators, we develop a variational problem theory by presenting necessary conditions of optimization for different kind of problems. An integration by parts formula is also proven, fundamental for the developing of our theory.

INTRODUCTION

In this work we combine two known ideas: fractional derivatives/integrals with respect to another function and variable fractional order derivatives/integrals (here, the order is given by a function $\gamma : [a,b]^2 \to (0,1)$), by considering the following fractional operators. For what concerns the fractional integrals, we consider the following ones

$$(left) \quad \mathbb{I}^{\gamma}_{a+}u(t) = \int_a^t \frac{1}{\Gamma(\gamma(t,s))} g'(s)(g(t)-g(s))^{\gamma(t,s)-1}u(s)\,ds,$$

$$(right) \quad \mathbb{I}^{\gamma}_{b-}u(t) = \int_t^b \frac{1}{\Gamma(\gamma(s,t))} g'(s)(g(s)-g(t))^{\gamma(s,t)-1}u(s)\,ds.$$

For fractional derivatives, we have the the Riemann–Liouville derivatives

$$(left) \quad \mathbb{D}^{\gamma}_{a+}u(t) = \frac{1}{g'(t)}\frac{d}{dt}\mathbb{I}^{1-\gamma}_{a+}u(t) = \frac{1}{g'(t)}\frac{d}{dt}\int_a^t \frac{1}{\Gamma(1-\gamma(t,s))} g'(s)(g(t)-g(s))^{-\gamma(t,s)}u(s)\,ds,$$

$$(right) \quad \mathbb{D}^{\gamma}_{b-}u(t) = \frac{-1}{g'(t)}\frac{d}{dt}\mathbb{I}^{1-\gamma}_{b-}u(t) = \frac{-1}{g'(t)}\frac{d}{dt}\int_t^b \frac{1}{\Gamma(1-\gamma(s,t))} g'(s)(g(s)-g(t))^{-\gamma(s,t)}u(s)\,ds,$$

and the Caputo derivatives

$$(left) \quad {}^C\mathbb{D}^{\gamma}_{a+}u(t) = \mathbb{I}^{1-\gamma}_{a+}\left[\frac{1}{g'(t)}\frac{du}{dt}(t)\right] = \int_a^t \frac{1}{\Gamma(1-\gamma(t,s))}(g(t)-g(s))^{-\gamma(t,s)}u'(s)\,ds,$$

$$(right) \quad {}^C\mathbb{D}^{\gamma}_{b-}u(t) = \mathbb{I}^{1-\gamma}_{b-}\left[\frac{-1}{g'(t)}\frac{du}{dt}(t)\right] = \int_t^b \frac{-1}{\Gamma(1-\gamma(s,t))}(g(s)-g(t))^{-\gamma(s,t)}u'(s)\,ds.$$

The variational problem considered is described next: minimize the functional

$$F(u) := \int_a^b L(t, u(t), {}^C\mathbb{D}^{\gamma}_{a+}u(t))\,dt \to \min,$$

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where $L : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and the functional is defined on the set

$$\Delta := \left\{ u \in C^1[a,b] : {}^C\mathbb{D}^\gamma_{a+} u \, \text{exists and is continuous on } [a,b] \right\}.$$

Define

$$[u](t) := (t, u(t), {}^C\mathbb{D}^\gamma_{a+} u(t)).$$

The necessary condition that allows to determine such optimal solutions is given by the next theorem:

Theorem 0.1. *Let $u^\star \in \Delta$ be an optimal solution for the variational problem. Then, the following Euler–Lagrange equation is met:*

$$\frac{\partial L}{\partial u}[u^\star](t) + g'(t) \mathbb{D}^\gamma_{b-} \left(\frac{\frac{\partial L}{\partial {}^C\mathbb{D}^\gamma_{a+} u}[u^\star](t)}{g'(t)} \right) = 0, \quad \forall t \in [a,b]. \tag{1}$$

Also, for arbitrary $u(a)$ holds

$$\mathbb{I}^{1-\gamma}_{b-} \frac{\frac{\partial L}{\partial {}^C\mathbb{D}^\gamma_{a+} u}[u^\star](t)}{g'(t)} = 0, \tag{2}$$

at $t = a$. For arbitrary $u(b)$, formula (2) holds at $t = b$.

References

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