A VARIATIONAL APPROACH FOR POLYNOMIAL FITTING ON MANIFOLDS

LUÍS MACHADO

Abstract. We address the problem of finding a polynomial curve that best fits a given data set of time-labelled points on a Riemannian manifold. The main drawback for defining the problem is the lack of explicit expressions for polynomials on manifolds. To overcome this issue, we propose a variational approach and derive the corresponding Euler-Lagrange equations. Due to the high nonlinearity of the Euler- Lagrange equations, we also propose a numerical optimization approach to obtain solutions for the problem. Numerical simulations will be provided in some specific Riemannian manifolds.

Introduction

The classical least squares method has been used for the first time in 1809, when Gauss and Legendre were predicting the planetary motion. In this classical setting, it is given a set of $k+1$ points, $p_0, \ldots, p_k$, in $\mathbb{R}^n$, a set of $k+1$ instants of time $0 = t_0 < \cdots < t_k = 1$, and the objective is to find a polynomial curve of degree m ($m \leq k$)

$$\begin{array}{rccl} \gamma : & [0,1] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & \gamma(t) = a_0 + a_1 t + \cdots + a_m t^m, \end{array}$$

that yields the minumum value for the functional

$$E(\gamma) = \sum_{i=0}^{k} d^2(p_i, \gamma(t_i)),$$

where d stands for the Euclidean distance. It is easy to prove the existence and uniqueness of the solution for this classical problem.

Our goal is to generalize this classical problem in the more general context of manifolds. However, in general, no explicit formulas for polynomials on manifolds are available. Following previous works [?] and [?], where polynomials on manifolds have been defined as the extremals of a certain functional, we proposed in [?] the following variational problem on a complete Riemannian manifold M:

$$(\mathcal{P}) \qquad \min_{\gamma \in \mathcal{C}} \frac{1}{2} \sum_{i=0}^{k} d^2\big(p_i, \gamma(t_i)\big) + \frac{\lambda}{2} \int_0^1 \left\langle \frac{D^m \gamma}{dt^m}, \frac{D^m}{dt^m} \right\rangle dt,$$

where d is now the geodesic distance in M, $\frac{D}{dt}$ denotes the covariant derivative with respect to the Levi Civita connection in M, λ denotes a positive real parameter and where $\mathcal{C}$ is the family of curves $\gamma : [0,1] \to M$ of class C^{m-1}, such that the restriction of γ to each

This work has been supported by Fundação para a Ciência e Tecnologia (FCT) under the project UIDB/00048/2020.

subinterval $[t_{i-1}, t_i]$ is smooth, for $i = 1, \ldots, k$. Moreover, it is assumed that $\dfrac{D^j\gamma}{dt^j}(t_i^+)$ and $\dfrac{D^j\gamma}{dt^j}(t_i^-)$ exist for all $j \geq m$ and $i = 0, \ldots, k$.

Using appropriate tools from the calculus of variations on Riemannian manifolds, the Euler-Lagrange equations are obtained in the next result.

Theorem 0.1. *A necessary condition for γ to be a solution for $(\mathcal{P})$ is that $\gamma \in C^{2m-2}([0,1])$, satisfies*

$$\frac{D^{2m}\gamma}{dt^{2m}} + \sum_{j=2}^{m}(-1)^j R\Big(\frac{D^{2m-j}\gamma}{dt^{2m-j}}, \frac{D^{j-1}\gamma}{dt^{j-1}}\Big)\frac{d\gamma}{dt} = 0, \ \forall t \in [t_{i-1}, t_i], \ (i = 1, \ldots, k),$$

and

$$\frac{D^j\gamma}{dt^j}(t_i^+) - \frac{D^j\gamma}{dt^j}(t_i^-) = \begin{cases} 0 & j = 1, \ldots, m-1 & (i = 1, \ldots, k-1) \\ 0 & j = m, \ldots, 2m-2 & (i = 0, \ldots, k) \\ \vdots & \vdots & \vdots \\ \frac{(-1)^m}{\lambda}\exp^{-1}_{\gamma(t_i)}(p_i) & j = 2m-1 & (i = 0, \ldots, k), \end{cases}$$

where $\exp_q^{-1}(p)$ denotes the velocity vector of the minimizing geodesic connecting points p and q (oriented from p to q).

We can deduce some properties directly from the above Euler-Lagrange equations. Namely, the geometric mean and the geodesic that best fits the given data arise as limiting processes of problem $(\mathcal{P})$.

In order to obtain approximate solutions for problem $(\mathcal{P})$, we closely follow the approach given in [?] for the Euclidean sphere. This numerical optimization procedure consists in the discretization of the functional defined in $(\mathcal{P})$, where geometric finite differences are used to approximate the covariant derivatives [?]. As an example, let us consider the case when $m = 2$. The forward, central and backward geometric differences are given respectively by

$$\begin{aligned} \frac{D^2\gamma}{dt^2}(t_0) &\approx \tfrac{1}{h^2}\big(\exp^{-1}_{\gamma(t_0)}(\gamma(t_2)) - 2\exp^{-1}_{\gamma(t_0)}(\gamma(t_1))\big) \\ \frac{D^2\gamma}{dt^2}(t_i) &\approx \tfrac{1}{h^2}\big(\exp^{-1}_{\gamma(t_i)}(\gamma(t_{i+1})) + \exp^{-1}_{\gamma(t_i)}(\gamma(t_{i-1}))\big) \\ \frac{D^2\gamma}{dt^2}(t_k) &\approx \tfrac{1}{h^2}\big(\exp^{-1}_{\gamma(t_k)}(\gamma(t_{k-2}) - 2\exp^{-1}_{\gamma(t_k)}(\gamma(t_{k-1}))\big). \end{aligned}$$

Numerical illustrations for different values of m (order of covariant derivative) and different number of points will be given in some specific Riemannian manifolds.

References

[1] N. Boumal and P.-A. Absil. A discrete regression method on manifolds and its application to data on $SO(n)$ *IFAC Proceedings Volumes*, 18th IFAC World Congress, 44 (2011), no. 1, 2284–2289, `https://doi.org/10.3182/20110828-6-IT-1002.00542`

[2] M. Camarinha; F. Silva Leite; P. Crouch. Splines of class C^k of non-Euclidean spaces. *IMA J. Math. Control Inform.*, 12 (1995), no. 4, 399–410, `https://doi.org/10.1093/imamci/12.4.399`

[3] L. Machado, L.; M. Teresa T. Monteiro. A numerical optimization approach to generate smoothing spherical splines. *J. Geom. Phys.*, 111 (2017), 71–81, `https://doi.org/10.1016/j.geomphys.2016.10.007`

[4] L. Machado; F. Silva Leite; K. Krakowski. Higher-order smoothing splines versus least squares problems on Riemannian manifolds. *J. Dyn. Control Syst*, 16 (2010), no. 1, 121–148, https://doi.org/10.1007/s10883-010-9080-1

[5] L. Noakes; G. Heinzinger; B. Paden. Cubic splines on curved spaces. *IMA J. Math. Control Inform.*, 6 (1989), no. 4, 465–473, https://doi.org/10.1093/imamci/6.4.465

Luís Machado; Institute of Systems and Robotics – University of Coimbra, Rua Silvio Lima - Polo II, 3030-290 Coimbra, Portugal & Department of Mathematics – University of Trás-os-Montes e Alto Douro (UTAD), Quinta de Prados, 5000-801 Vila Real, Portugal

Email address: lmiguel@utad.pt