VARIATIONAL PRINCIPLE OF HERGLOTZ FOR A NEW CLASS OF PROBLEMS WITH DEPENDENCE ON THE FREE ENDPOINT CONDITIONS AND A REAL PARAMETER

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Abstract. The main goal of this talk is to generalize the variational problem of Herglotz considering the case where the Lagrangian depends not only on the independent variable, an unknown function x and its derivative, and an unknown functional z, but also on the end points conditions and a real parameter. Herglotz's problems of calculus of variations of this type cannot be solved using the standard theory. Since the variational problem of Herglotz is a generalization of the classical problem of the calculus of variations, from our main results a number of important corollaries are obtained. Main results of this talk are necessary optimality condition of Euler- Lagrange type, natural boundary conditions and the DuBois-Reymond condition for our non-standard variational problem of Herglotz type.

Introduction

In this talk we will extend the variational problem of Herglotz type ([1], [2], [4]), to the case where the Lagrangian L may depend on the states values $x(a)$ and $x(b)$ and also on a real parameter ζ. More specifically, we will study the following non-standard Herglotz type problem:

Problem ($\mathcal{P}$): Determine $x \in C^2([a,b],\mathbb{R})$, $z \in C^1([a,b],\mathbb{R})$ and $\zeta \in \mathbb{R}$ such that

$$z(b) \longrightarrow extr$$

where (x, z, ζ) is such that

$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), z(t), x(a), x(b), \zeta\right), \quad t \in [a,b], \tag{1}$$

and

$$z(a) = \gamma, \tag{2}$$

for a given $\gamma \in \mathbb{R}$, and L satisfies the following conditions:

(1) $(u_0, u_1, u_2, u_3, u_4, u_5) \to L(t, u_0, u_1, u_2, u_3, u_4, u_5)$ is a C^1 function for any $t \in [a,b]$;
(2) functions $t \to \partial_i L\left(t, x\left(t\right), \dot{x}\left(t\right), z\left(t\right), x\left(a\right), x\left(b\right), \zeta\right)$, are differentiable for all $i = 2, \ldots, 7$ and for all admissible triplet (x, z, ζ).

The following result provides necessary conditions for an admissible triplet (x, z, ζ) to be a local extremizer of $z(b)$, where (x, z, ζ) satisfies (1) and (2).

Work supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), within project UIDB/04106/2020.

Theorem 0.1. [3] *If (x,z,ζ) is a solution of the non-standard Herglotz's variational problem $(\mathcal{P})$, then (x,z,ζ) satisfies the generalized Euler-Lagrange equation*

$$\partial_2 L\left[x,z,\zeta\right](t)+\partial_4 L\left[x,z,\zeta\right](t)\cdot\partial_3 L\left[x,z,\zeta\right](t)-\frac{d}{dt}\partial_3 L\left[x,z,\zeta\right](t)=0,\quad t\in[a,b], \tag{3}$$

and the condition

$$\int_a^b \lambda(t)\cdot\partial_7 L\left[x,z,\zeta\right](t)\,dt=0, \tag{4}$$

where

$$\lambda(t)=e^{-\int_a^t \partial_4 L[x,z,\zeta](s)ds}. \tag{5}$$

Moreover,

(1) *if $x(a)$ is undetermined, then (x,z,ζ) satisfies the following condition*

$$\partial_3 L\left[x,z,\zeta\right](a)=\frac{1}{\lambda(a)}\int_a^b \lambda(t)\cdot\partial_5 L\left[x,z,\zeta\right](t)\,dt; \tag{6}$$

(2) *if $x(b)$ is undetermined, then (x,z,ζ) satisfies the following condition*

$$\partial_3 L\left[x,z,\zeta\right](b)=-\frac{1}{\lambda(b)}\int_a^b \lambda(t)\cdot\partial_6 L\left[x,z,\zeta\right](t)\,dt. \tag{7}$$

Remark 0.2. Observe that:

(1) from Theorem 0.1 we can deduce as corollaries some well known results from the classical calculus of variations and from the standard Herglotz's variational calculus;
(2) the two non-standard natural boundary conditions of Theorem 0.1 clearly shows that the standard and non-standard Herglotz's variational problems are of a different nature;
(3) in the basic problem of the classical calculus of variations, the Lagrangian L does not depend on $x(a)$, $x(b)$, z and ζ. In this case, (6) and (7) reduce to the well known natural boundary conditions: $\partial_3 L(a,x(a),\dot{x}(a))=0$ and $\partial_3 L(b,x(b),\dot{x}(b))=0$.

Theorem 0.3. [3] *If (x,z,ζ) is a solution of problem $(\mathcal{P})$, then (x,z,ζ) satisfies the equation*

$$\frac{d}{dt}\Big[\lambda(t)\cdot\Big(L\left[x,z,\zeta\right](t)-\dot{x}(t)\,\partial_3 L\left[x,z,\zeta\right](t)\Big)\Big]=\lambda(t)\cdot\partial_1 L\left[x,z,\zeta\right](t), \tag{8}$$

for all $t\in[a,b]$.

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